

The Fundamental Law of Portfolio Enhancement: A Unified Performance Budget for Portfolio Returns and Characteristics

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Abstract

We generalize the Fundamental Law of Active Management to hybrid mandates combining stochastic alpha with deterministic objectives, e.g. carbon abatement. We derive the Fundamental Law of Portfolio Enhancement, decomposing the maximum "Enhancement Ratio" into skill, characteristic targets, and a novel interaction term capturing the alignment between alpha and the hybrid mandate. Our framework reveals that the tracking-error cost of a characteristic's deterministic tilt is determined not only by breadth and return precision, but also by the characteristic's cross-sectional dispersion. We quantify constraint drag through a generalized transfer coefficient. Using 2015-2024 data and out-of-sample optimizations for large-cap US, UK and Australian universes, we empirically substantiate the conceptual development of the FLPE, providing a robust basis for pricing sustainability constraints in active portfolios.

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1 Introduction

Benchmark-relative portfolio management is increasingly multi-objective ¹. Many mandates now pair the goal of risk-adjusted active return with explicit portfolio-characteristic targets, such as carbon intensity, ESG scores, industry exclusions or factor tilts, while remaining within a tracking-error budget and a practical constraint set (long-only, bounds, turnover, sector limits). To illustrate, suppose a manager is asked to reduce benchmark-relative carbon intensity by 20% *at the time of the optimization*. In some universes this is achievable with little tracking error sacrifice because carbon exposures are dispersed and close substitutes exist. In others, the same target requires concentrated benchmark deviations and substantial tracking error. Under long-only and weight bounds, marginal tracking error costs often rise sharply at ambitious targets.

This study focuses on the following questions: what determines the slope and curvature of the enhancement–tracking-error trade-off, and why does it vary across asset universes and constraints? Carbon abatement is our leading case study, but the framework applies to any characteristic that can be measured at portfolio formation. The goal is not to replace mean–variance logic, but to express it in a performance-budget form that is operational for hybrid mandates.

Grinold’s Fundamental Law of Active Management (FLAM) (Grinold (1989)) links an unconstrained strategy’s risk -adjusted return (measured by the information ratio, IR) to the manager’s forecasting skill (measured by the information coefficient, IC) and to the number of independent bets the manager makes (breadth). Clarke, Silva, and Thorley (2002) introduced the transfer coefficient (TC) to quantify how constraints reduce the translation of forecasts into active weights, therefore acting as a performance drag. Clarke, Silva, and Thorley (2006) generalized the FLAM to a fully populated covariance matrix and clarified the interpretation of breadth under correlated residual returns. Menchero (2017) further

¹See for instance Blitz et al. (2024), De Nard, Engle, and Kelly (2024), and Bolton, Kacperczyk, and Samana (2024) and Matos (2020).

refined the role of correlations and transfer coefficients using an exact correlation-based representation. Ding and Martin (2017) provided a factor-model foundation for the FLAM that relaxes the restrictive assumptions of constant IC and zero factor risk in alpha forecasts, nesting previously proposed formulations of the law.

These contributions are highly effective for portfolio problems where the objective is stochastic and signals lead to imperfect forecasts. They are less applicable for objectives that are known at the time of portfolio formation (e.g., carbon intensity) and do not require forecasts of future outcomes. If one treats a deterministic characteristic as a perfect forecast and sets $IC \approx 1$, the classic FLAM and its extensions can imply implausibly large performance as breadth grows. More importantly, it does not provide interpretable drivers of the tracking-error cost of changing a characteristic, nor does it explain cross-market differences in those costs.

In this paper, we develop the Fundamental Law of Portfolio Enhancement (FLPE), a unified performance budget for benchmark-relative portfolios that pursue (i) stochastic objectives based on return forecasts (alpha) and (ii) deterministic objectives based on known characteristics. The FLPE is derived within the standard quadratic optimization / tracking-error geometry and is built on the rich factor-model structure of Ding and Martin (2017). It delivers an “enhancement ratio” (ER)—enhancement per unit tracking error—analogous to the IR, and provides a decomposition that is economically interpretable for hybrid mandates.

Three key features of our conceptual development are worth mentioning from the outset. First, the FLPE nests established results in the expected-return setting. When the objective is purely stochastic (alpha only), the FLPE collapses to the familiar FLAM expressions and is consistent with existing tracking-error constrained mean–variance results (e.g., Jorion (2003)) and with the fully populated covariance generalization emphasized by Clarke, Silva, and Thorley (2006). Second, the FLPE yields an interpretable slope for deterministic enhancements. In the purely deterministic case, and under simplifying assumptions used only

for intuition (e.g., a diagonal approximation to residual covariance and no binding weight constraints), achievable enhancement is approximately proportional to tracking error times the square root of three quantities: effective breadth, an average precision term (linked to the inverse of residual variances), and the cross-sectional dispersion of the enhancement signal. This dispersion term is economically central and, to our knowledge, novel: if the characteristic is nearly constant across securities, portfolios cannot change it without extreme benchmark deviation; as dispersion increases, the same target becomes cheaper to implement. Third, the FLPE clarifies hybrid mandates and interactions among objectives. In the general case the squared ER decomposes into an “active” term (alpha), a “static” term (deterministic enhancement), and a "cross term" capturing alignment or conflict between alpha and the characteristic. This cross term is also novel to the FLAM literature and formalizes when a characteristic mandate is likely to be inexpensive because it tilts in the same direction as alpha, and when it crowds out alpha by forcing active weights away from the return forecast.

In an unconstrained quadratic setting the enhancement–tracking-error relation is linear, which is a standard efficient-frontier result. The practical questions are therefore what determines the slope of the unconstrained trade-off and what drives the curvature observed under realistic constraints. To connect these, we extend the FLPE to constrained portfolios using a transfer-coefficient style representation: constraints reduce attainable enhancement by limiting how closely the feasible portfolio can track the unconstrained optimum. This connects the curvature of observed trade-off curves to the interaction of the signal with the constraint set, in the spirit of Clarke, Silva, and Thorley (2002) and Menchero (2017).

We illustrate the framework using carbon intensity and provide out-of-sample evidence for large -cap universes across Australia, the United States, and the United Kingdom based on data from January 2015 to December 2024. Each month we compute the FLPE components and examine how they map into realized enhancement–tracking-error relations under uncon-

strained and constrained implementations. This time-series design addresses stability and robustness: whether the slope drivers and the constraint effects vary materially over time and across markets.

For researchers and practitioners, the FLPE provides a transparent toolkit for performance budgeting and mandate design. It supports ex ante comparison of the tracking-error cost of the same enhancement target across universes, helps diagnose whether changes in “difficulty” reflect residual risk versus characteristic dispersion, and clarifies when enhancement objectives are naturally aligned with alpha. More broadly, it disciplines discussions of cross-market differences by tying them to observable inputs rather than to ad hoc conjectures.

The remainder of the paper proceeds as follows. Section 2 defines enhancement, tracking-error risk, and the optimization setting. Section 3 derives the unconstrained FLPE and its decomposition. Section 4 introduces portfolio constraints and relates them to transfer coefficients. Section 5 derives the constrained FLPE. Section 6 presents the empirical analysis and carbon case study. Section 7 concludes.

2 A Unified Model Framework

2.1 The Core Objective: Enhancement

We define the manager’s total enhancement vector $\mathbf{e}_t$ as a weighted combination of a stochastic return objective ($\mathbf{r}_t$) and a deterministic characteristic objective ($\mathbf{d}_t$):

$$\mathbf{e}_t = a\mathbf{r}_t + b\mathbf{d}_t \tag{1}$$

where a and b are non-negative scalar preferences, $\mathbf{r}_t$ is the $N \times 1$ vector of (stochastic) active residual returns, and $\mathbf{d}_t$ is the $N \times 1$ vector of (deterministic) characteristics, known at time $t - 1$. Of course, we do not know $\mathbf{r}$ at the time of portfolio formation, and instead we use Alpha which is the estimate of $\mathbf{r}$ based on the information available in the previous

time period, $\alpha_t = E(\mathbf{r}_t | I_{t-1})$. However we do know $\mathbf{d}$ at the time of portfolio formation, as it is simply the known characteristics of the portfolio at that time, such as for instance its Quality score.

2.2 Defining the Model Components

Please note that much of this section follows directly from Ding and Martin (2017).

Definition 1 (Residual Returns). *The **residual return** vector, $\mathbf{r}_t$, represents the $N \times 1$ vector of asset returns in excess of a benchmark, after adjusting for systematic benchmark beta. For a given asset i , $r_{it} = r_{i,t}^{Total} - \beta_i r_{Benchmark,t}$ where $r_{i,t}^{Total}$ is the total return in excess of the risk free rate. We assume that these returns have an unconditional mean of zero: $E[\mathbf{r}_t] = \mathbf{0}$*

Definition 2 (Standardized Residual Returns). *The **standardized residual return** vector, $\tilde{\mathbf{r}}_t$, is the vector of residual returns scaled by their conditional volatility. Let $\Lambda_t = \text{diag}(\sigma_{r_{it}}^2)$ be the $N \times N$ diagonal matrix of conditional variances. Then the standardised residuals are calculated as $\tilde{\mathbf{r}}_t = \Lambda_t^{-1/2} \mathbf{r}_t$.*

The return-generating process is assumed to follow a K -factor model

$$\tilde{\mathbf{r}}_t = Z_{t-1} \mathbf{f}_t + \boldsymbol{\epsilon}_t \quad (2)$$

where:

- Z_{t-1} is the $N \times K$ matrix of **factor exposures** (the "signals"), known at $t - 1$. They may be "standardized firm specific quantities such as book-to-market, earnings-to-price, firm size..."
- $\mathbf{f}_t$ is the $K \times 1$ random vector of **factor returns** which turn out to be "Information Coefficients".
- $\boldsymbol{\epsilon}_t$ is the $N \times 1$ vector of idiosyncratic errors.

From this model, we derive the manager's alpha forecast:

Definition 3 (Alpha Vector). *The **alpha vector**, α_t , is the manager's ex-ante forecast of residual returns, conditional on the information I_{t-1} (which includes Z_{t-1}) (Ding and Martin 2017, Section 2.2, equation 6).*

$$\alpha_t \equiv E[\mathbf{r}_t | I_{t-1}] = E[\Lambda_t^{1/2} \tilde{\mathbf{r}}_t | I_{t-1}] = \Lambda_t^{1/2} E[Z_{t-1} \mathbf{f}_t + \boldsymbol{\epsilon}_t | I_{t-1}] \quad (3)$$

Assuming $E[\boldsymbol{\epsilon}_t | I_{t-1}] = \mathbf{0}$, this simplifies to the factor-model specific form from Ding and Martin (2017, Section 4.1, equation 37):

$$\alpha_t = \Lambda_t^{1/2} Z_{t-1} E[\mathbf{f}_t | I_{t-1}] = \Lambda_t^{1/2} Z_{t-1} \mathbf{f} \quad (4)$$

where $\mathbf{f} = E[\mathbf{f}_t]$ is the $K \times 1$ vector of mean factor returns, representing the manager's **persistent skill**.

2.2.1 Deterministic Objectives

We now model the deterministic objective. We assume there exists a universe of M observable characteristics for each of the N assets (e.g., 'Value', 'Momentum', 'ESG', 'Carbon Intensity',...) represented by an $N \times M$ matrix X_{t-1} .

Definition 4 (Deterministic Objective). *The **deterministic objective vector**, $\mathbf{d}_t$, is an $N \times 1$ vector representing a desired, known-at- $t - 1$ portfolio characteristic. We model this as a fixed linear combination of the M available characteristics:*

$$\mathbf{d}_t = X_{t-1} \mathbf{c}_d \quad (5)$$

where $\mathbf{c}_d$ is an $M \times 1$ vector of **deterministic tilt weights**. For example, if the manager wishes to tilt towards the 3rd characteristic (e.g., 'ESG') and against the 5th (e.g., 'Carbon'),

$\mathbf{c}_d$ would have a positive value at index 3 and a negative value at index 5 ².

In the empirical section of this paper, we specialize this general definition to the case of carbon abatement. There, $M = 1$, and the matrix X_{t-1} reduces to the vector of security-level carbon intensities. The weight vector $\mathbf{c}_d$ becomes a scalar representing the direction of the tilt (abatement).

More generally, the framework explicitly allows for the three cases:

- **FLAM-only factors:** Columns in Z_{t-1} that are not in (and are uncorrelated with) X_{t-1} .
- **Deterministic-only factors:** Columns in X_{t-1} that are not in (and are uncorrelated with) Z_{t-1} .
- **Cross-term factors:** Factors that are columns in (or correlated with) *both* Z_{t-1} and X_{t-1} .

2.2.2 The Risk Model

Definition 5 (Risk Matrix). *The **risk matrix**, Ω_t , is the conditional covariance matrix of the forecast errors, $\mathbf{r}_t - \alpha_t$.*

$$\Omega_t \equiv E[(\mathbf{r}_t - \alpha_t)(\mathbf{r}_t - \alpha_t)' | I_{t-1}] \quad (6)$$

As derived in Ding and Martin (2017), this is:

$$\Omega_t = \Lambda_t^{1/2} (Z_{t-1} \Sigma_{\mathbf{f}} Z_{t-1}' + \sigma_\epsilon^2 I) \Lambda_t^{1/2} \quad (7)$$

²This set up does not explicitly allow for enhancements set as constraints. They can of course be dealt with via the transfer coefficient discussed in a later section, but could be better dealt with via solving the re-framed optimization problem where they are included as explicit constraints.

where $\Sigma_{\mathbf{f}} = \text{Cov}(\mathbf{f}_t)$ is the $K \times K$ covariance matrix of factor returns, known as **strategy risk**, and σ_ϵ^2 is the idiosyncratic error variance. The strategy risk is related to the volatility in the information coefficient itself, where a higher volatility will result in a lower IR.

2.3 The Optimization Problem

The manager seeks to find the $N \times 1$ vector of active weights, $\Delta \mathbf{w}_t$, that maximizes utility.

The combined objective vector is:

$$\boldsymbol{\mu}_{obj,t} \equiv E[\mathbf{e}_t | I_{t-1}] = a\alpha_t + b\mathbf{d}_t \quad (8)$$

The utility function, following the quadratic utility setup is:

$$\max_{\Delta \mathbf{w}_t} U(\Delta \mathbf{w}_t) = E[\Delta \mathbf{w}_t' \mathbf{e}_t | I_{t-1}] - \lambda \text{Var}(\Delta \mathbf{w}_t' \mathbf{r}_t | I_{t-1}) \quad (9)$$

Substituting our defined terms:

$$\max_{\Delta \mathbf{w}_t} U(\Delta \mathbf{w}_t) = \Delta \mathbf{w}_t' \boldsymbol{\mu}_{obj,t} - \lambda (\Delta \mathbf{w}_t' \Omega_t \Delta \mathbf{w}_t) \quad (10)$$

This maximization is subject to a set of constraints, $\mathcal{C}$, which at a minimum includes $\Delta \mathbf{w}_t' \mathbf{1} = 0$, but can also include long-only, turnover, or factor-exposure constraints.

3 The Unconstrained FLPE Framework

First, we analyze the *fully unconstrained* case, where $\mathcal{C}$ is empty. This gives us the theoretical maximum *ER* achievable, which we call ER_U .

Definition 6 (Enhancement Ratio). *The **Enhancement Ratio** (ER) is a generalization of the Information Ratio to the case where the objective includes not only expected return, but also portfolio characteristics. It is the ratio of expected Enhancement to the active risk*

(tracking error) incurred by that enhancement. The squared ER is denoted as $G(\Delta \mathbf{w}_t)$:

$$G(\Delta \mathbf{w}_t) = \frac{(\Delta \mathbf{w}_t' \boldsymbol{\mu}_{obj,t})^2}{\Delta \mathbf{w}_t' \Omega_t \Delta \mathbf{w}_t} \quad (11)$$

As shown in Appendix A, the unconstrained portfolio $\mathbf{h}_u$ that maximizes utility (equation 10) is $\mathbf{h}_u \propto \Omega_t^{-1} \boldsymbol{\mu}_{obj,t}$, and this portfolio also maximizes the ER. The maximized squared ER for this unconstrained portfolio, G_U , can be calculated by substituting the definition of $\mathbf{h}_u$ into equation (11), and is:

$$G_U = \boldsymbol{\mu}_{obj,t}' \Omega_t^{-1} \boldsymbol{\mu}_{obj,t} \quad (12)$$

The **Unconstrained Fundamental Law of Portfolio Enhancement** is the unconditional expectation of this quantity, $E[G_U]$. We substitute our full objective vector $\boldsymbol{\mu}_{obj,t} = a\alpha_t + b\mathbf{d}_t$ and show (see Appendix F and Appendix G) that

$$\begin{aligned} E[G_U] &= E[(a\alpha_t + b\mathbf{d}_t)' \Omega_t^{-1} (a\alpha_t + b\mathbf{d}_t)] \\ &= \underbrace{a^2 E[\alpha_t' \Omega_t^{-1} \alpha_t]}_{\text{Active Contribution}} + \underbrace{b^2 E[\mathbf{d}_t' \Omega_t^{-1} \mathbf{d}_t]}_{\text{Static Contribution}} + \underbrace{2ab E[\alpha_t' \Omega_t^{-1} \mathbf{d}_t]}_{\text{Cross-Term}} \end{aligned} \quad (13)$$

This is the full *unconstrained* performance budget. We now analyze each term.

3.1 Analysis of the Unconstrained FLPE Components

3.1.1 Term 1: The Active Contribution

In Appendix F we show that $E[G_{\text{Active}, U}] = a^2 E[\alpha_t' \Omega_t^{-1} \alpha_t]$. The active contribution represents the Pure Skill component of the performance budget. The core quantity $\alpha_t' \Omega_t^{-1} \alpha_t$ is mathematically identical to the squared maximum Information Ratio (IR^2) available to an unconstrained manager. This is discussed at length in existing treatments of the classical FLAM and its extensions (see, among others, Ding and Martin (2017)).

3.1.2 Term 2: The Static Contribution

The unconstrained static contribution is $E[G_{\text{Static}, U}] = b^2 E[\mathbf{d}'_t \Omega_t^{-1} \mathbf{d}_t]$. This term can be interpreted as a Mahalanobis Distance. To clarify:

- **Definition:** The squared Mahalanobis distance of a vector $\mathbf{x}$ from the origin, relative to a covariance matrix S , is $D_M^2(\mathbf{x}) = \mathbf{x}' S^{-1} \mathbf{x}$.
- **Intuition:** It is a risk-adjusted measure of size. Think of it as a multivariate "z-score" squared. A vector $\mathbf{x}$ pointing in a direction of high portfolio risk (a high-variance eigenvector of S) is not "unusual" and its Mahalanobis distance will be small. Conversely, a vector $\mathbf{x}$ pointing in a very low-risk direction (a low-variance eigenvector) is highly "unusual" and will have a large Mahalanobis distance.
- **Application:** The static term $E[\mathbf{d}'_t \Omega_t^{-1} \mathbf{d}_t]$ is, thus, the expected squared Mahalanobis distance of the deterministic objective vector $\mathbf{d}_t$, measured against the conditional risk matrix Ω_t .
- **Interpretation:** This term measures the "risk-adjusted size" of our deterministic tilt. A large $G_{\text{Static}, U}$ value means we can achieve a large deterministic tilt (e.g., a big improvement in ESG score) for a small amount of active risk (Ω_t).

3.1.3 Term 3: The Cross-Term Contribution

This is our primary term of interest. In Appendix G we show that

$$E[G_{\text{Cross}, U}] = 2ab E[\alpha'_t \Omega_t^{-1} \mathbf{d}_t] = 2ab \mathbf{f}' \Sigma_{Z\Omega X} \mathbf{c}_d, \quad (14)$$

where $\Sigma_{Z\Omega X} \equiv E[Z'_{t-1} \Lambda_t^{1/2} \Omega_t^{-1} X_{t-1}]$.

The cross-term, $E[G_{\text{Cross}, U}]$, captures the interaction between the active alpha model and the deterministic objective. It is a bilinear form that "bridges" the manager's skill vector ($\mathbf{f}$) with their deterministic tilt vector ($\mathbf{c}_d$). The bridge itself is the matrix $\Sigma_{Z\Omega X}$, which we

define as "**expected risk-adjusted cross-covariance**". To gain some intuition, we break down the latter into three concepts:

- **Cross-Covariance:** The matrix $(Z'X)$ is, fundamentally, a measure of the relationship between the alpha factors Z and the characteristic factors X . Note that both Z and X refer to firm characteristics: in Z they are normalized characteristics believed to have correlations to returns, and in X they are characteristics targeted for a particular non-stochastic portfolio tilt. An element $(Z'X)_{km}$ would be the simple cross-sectional covariance between alpha factor k and characteristic m .
- **Risk-Adjusted:** Our matrix, however, is not $E[Z'X]$. It is $E[Z'(\dots\Omega_t^{-1}\dots)X]$. The presence of the inverse risk matrix Ω_t^{-1} (the "precision" matrix) "risk-adjusts" this entire relationship. It measures the covariance between Z and X *in the space defined by the inverse risk metric*. A strong correlation between an alpha factor and a characteristic will be "downweighted" by Ω_t^{-1} if that correlation exists along a direction of high risk (a high-variance eigenvector of Ω_t). It measures the "effective" or "risk-relevant" overlap between the two factor sets.
- **Expected:** The expectation $E[\cdot]$ is taken over the joint distribution of all random variables (Z , X , Λ , etc.) to find the long-run average value of this complex, risk-adjusted relationship.

In short, this matrix quantifies the effective, risk-adjusted alignment between the manager's alpha-generating factors and their deterministic tilt characteristics.

3.1.4 Analysis of Synergy and Conflict (The Cross-Term Sign)

We now analyze the sign of the cross-term, $E[G_{\text{Cross, U}}] = 2ab\mathbf{f}'\Sigma_{ZX}\mathbf{c}_d$, to understand when the deterministic objective aids the active returns ("Synergy") and when it acts as a drag ("Conflict").

To build intuition, consider the simplest possible case: a single-factor world ($K = 1$) with

a single deterministic characteristic ($M = 1$). Furthermore, assume the manager's alpha signal is identical to the characteristic they are mandated to target (e.g., a "Value" manager mandated to maintain a high "Value" score).

- $Z_{t-1} = X_{t-1}$ is an $N \times 1$ vector $\mathbf{z}$.
- The skill vector $\mathbf{f}$ is a scalar IC (Information Coefficient).
- The tilt vector $\mathbf{c}_d$ is a scalar c_{tilt} .

Substituting these into the cross-term equation gives:

$$E[G_{\text{Cross, U}}] \approx 2ab \cdot IC \cdot c_{tilt} \cdot E \left[\mathbf{z}' \left(\Lambda_t^{1/2} \Omega_t^{-1} \right) \mathbf{z} \right] \quad (15)$$

The sign of this contribution depends on the signs of the scalar term $\mathcal{Q} = \mathbf{z}'(\Lambda_t^{1/2}\Omega_t^{-1})\mathbf{z}$ and on the sign of $IC \times c_{tilt}$. In Appendix H we show that, under mild and standard conditions, $\mathcal{Q}$ can be assumed to be positive

Intuition: Synergy vs. Conflict With the risk-adjusted overlap term ($\mathbf{z}'(\Lambda_t^{1/2}\Omega_t^{-1})\mathbf{z}$) established as positive, the sign of the contribution is determined by the product of the manager's skill and the mandated tilt:

$$\text{Sign}(G_{\text{Cross}}) \propto \text{Sign}(IC \times c_{tilt}) \quad (16)$$

This reveals two distinct regimes for the active manager:

1. **Synergy** ($IC \times c_{tilt} > 0$): The manager has skill in the factor ($IC > 0$) and the mandate requires a long position in that factor ($c_{tilt} > 0$). *Interpretation:* The deterministic constraint forces the manager to hold positions they *would have held anyway* to generate alpha. The "constraint" essentially funds the alpha bets, freeing up the risk budget for other opportunities. The combined ER is greater than the sum of its

parts.

2. **Conflict** ($IC \times c_{tilt} < 0$): The manager has skill in the factor ($IC > 0$), but the mandate requires a negative tilt ($c_{tilt} < 0$). *Interpretation:* This is the "fighting the mandate" scenario. To generate alpha, the manager wants to go Long, but the mandate forces them to go Short. The manager must burn significant active risk just to neutralize the mandated short position before they can build the long position required for alpha. The cross-term is negative, acting as a severe drag on the total Enhancement Ratio.

4 Analysis of Constraints and the Generalized Transfer Coefficient

In Section 3, we derived the *unconstrained* FLPE. In that formulation, $E[G_U]$ represents the theoretical maximum performance a manager can achieve given their skill and objectives, assuming no constraints other than $\mathbf{1}'\Delta\mathbf{w} = 0$. However, all practical portfolio management is subject to a set of constraints: long-only, turnover limits, exposure limits, etc. (Clarke, Silva, and Thorley 2002; Ding and Martin 2017). These constraints force a manager to hold a portfolio, which we will call $\mathbf{h}_c$, that is suboptimal relative to the theoretical fully unconstrained portfolio, which we will call $\mathbf{h}_u$. This deviation inevitably leads to a reduction in the achievable ER . The classic FLAM framework, as discussed in Clarke, Silva, and Thorley (2002), accounts for this loss using a single scalar, the Transfer Coefficient (TC), which compares the actually-held portfolio $\mathbf{h}_c$ (which must obey all constraints, including $\mathbf{1}'\Delta\mathbf{w} = 0$) against the true theoretical optimum, $\mathbf{h}_u$. We will now extend this concept to our generalized framework.

4.1 The Unconstrained Benchmark

Before quantifying the effect of constraints, we explicitly derive the Enhancement Ratio of the theoretical unconstrained portfolio, $\mathbf{h}_u$. As established in Section 3 (prior to equation 12), the unconstrained active weights are proportional to the risk-adjusted objective:

$$\mathbf{h}_u = k\Omega_t^{-1}\boldsymbol{\mu}_{obj,t} \quad (17)$$

Substituting this into the definition of the squared Enhancement Ratio $G(\Delta\mathbf{w}_t)$ from equation (11) yields:

$$ER_U = \sqrt{G(\mathbf{h}_u)} = \frac{\mathbf{h}'_u\boldsymbol{\mu}_{obj,t}}{\sqrt{\mathbf{h}'_u\Omega_t\mathbf{h}_u}} \quad (18)$$

The numerator is:

$$\mathbf{h}'_u\boldsymbol{\mu}_{obj,t} = (k\Omega_t^{-1}\boldsymbol{\mu}_{obj,t})'\boldsymbol{\mu}_{obj,t} = k(\boldsymbol{\mu}'_{obj,t}\Omega_t^{-1}\boldsymbol{\mu}_{obj,t}) \quad (19)$$

Recognizing the term in parentheses as G_U from equation (12), the numerator is kG_U . The denominator is:

$$\begin{aligned} \sqrt{\mathbf{h}'_u\Omega_t\mathbf{h}_u} &= \sqrt{(k\Omega_t^{-1}\boldsymbol{\mu}_{obj,t})'\Omega_t(k\Omega_t^{-1}\boldsymbol{\mu}_{obj,t})} \\ &= \sqrt{k^2\boldsymbol{\mu}'_{obj,t}\Omega_t^{-1}\Omega_t\Omega_t^{-1}\boldsymbol{\mu}_{obj,t}} \\ &= k\sqrt{\boldsymbol{\mu}'_{obj,t}\Omega_t^{-1}\boldsymbol{\mu}_{obj,t}} = k\sqrt{G_U} \end{aligned} \quad (20)$$

The scaling factor k cancels out, confirming that the maximum achievable Enhancement Ratio is the square root of the quadratic form G_U :

$$ER_U = \frac{kG_U}{k\sqrt{G_U}} = \sqrt{G_U} \quad (21)$$

Following the framework of Clarke, Silva, and Thorley (2002) and Menchero (2017) we define

the Generalized Transfer Coefficient (GTC):

Definition 7 (Generalized Transfer Coefficient). *The GTC is the ratio of the constrained portfolio's Enhancement Ratio (ER_C) to the maximum achievable (i.e., unconstrained) Enhancement Ratio (ER_U).*

$$GTC = \frac{ER_C}{ER_U} \quad (22)$$

where:

- $ER_C = \frac{\mathbf{h}'_c \boldsymbol{\mu}_{obj}}{\sqrt{\mathbf{h}'_c \Omega_t \mathbf{h}_c}}$
- $ER_U = \sqrt{G_U} = \sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}}$

Substituting into (22) yields:

$$GTC = \frac{\left(\frac{\mathbf{h}'_c \boldsymbol{\mu}_{obj}}{\sqrt{\mathbf{h}'_c \Omega_t \mathbf{h}_c}} \right)}{\sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}}} = \frac{\mathbf{h}'_c \boldsymbol{\mu}_{obj}}{\sqrt{\mathbf{h}'_c \Omega_t \mathbf{h}_c} \sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}}} \quad (23)$$

This definition is a direct generalization of the TC defined in Menchero (2017) , with the enhancement vector $\boldsymbol{\mu}_{obj}$ replacing their alpha vector α .

4.2 GTC as a Correlation

Let:

- Let $r_{c,t} = \mathbf{h}'_c \mathbf{r}_t$ be the scalar active return of the constrained portfolio at time t .
- Let $r_{u,t} = \mathbf{h}'_u \mathbf{r}_t$ be the scalar active return of the unconstrained portfolio at time t .

As shown in Appendix E, it turns out that $GTC = \text{Corr}(r_c, r_u)$. Therefore, GTC measures the correlation between the active return streams generated by the two portfolios. It is *not* the correlation between the two $N \times 1$ vectors of weights. The measure is intuitive, as GTC is

meant to reflect how well the constrained portfolio's return stream tracks the return stream of the unconstrained portfolio.

5 The Complete Constrained FLPE

Finally, we take the expectation of $G_C = GTC^2 \cdot G_U$ and substitute our three-term expansion for G_U (from equation 13). Assuming that GTC is relatively stable and independent of G_U , we can approximate $E[G_C] \approx E[GTC^2] \cdot E[G_U]$. For simplicity, we treat GTC as a constant property of the constraints. It follows that the **Constrained Fundamental Law of Portfolio Enhancement** is:

$$E[G_C] \approx GTC^2 \times \left(\underbrace{a^2 E[G_{\text{Active}, U}]}_{\text{Active Term}} + \underbrace{b^2 E[G_{\text{Static}, U}]}_{\text{Static Term}} + \underbrace{2ab E[G_{\text{Cross}, U}]}_{\text{Cross Term}} \right) \quad (24)$$

where:

- $E[G_{\text{Active}, U}] \approx \mathbf{f}' \left(\frac{\sigma_\epsilon^2}{N} I + \Sigma_{\mathbf{f}} \right)^{-1} \mathbf{f}$
- $E[G_{\text{Static}, U}] \approx \mathbf{c}_d' \left(E[X'_{t-1} \Omega_t^{-1} X_{t-1}] \right) \mathbf{c}_d$
- $E[G_{\text{Cross}, U}] \approx \mathbf{f}' \Sigma_{Z\Omega X} \mathbf{c}_d$

This final, unified law demonstrates that real-world constraints act as a uniform "drag" (since $GTC^2 \leq 1$) on all three sources of portfolio enhancement.

5.1 Diagonal Approximation

Let us now analyze the case where there is *no* active alpha model, only a deterministic objective. Here $a = 0$ and without loss of generality we can let $b = 1$.

For our example, we will assume that the only static objective is a carbon abatement.

- The alpha model does not exist, so $Z = 0$, $\mathbf{f} = 0$, and $a = 0$.

- The deterministic objective is to tilt against carbon intensity, so $\mathbf{d}_t = \mathbf{c}_{I,t}$ (the vector of carbon intensities) and $b = 1$.
- The active risk matrix Ω_t is diagonal, $\Omega_t = \text{diag}(\sigma_i^2)$. This is a common assumption in the classic Grinold (1989) FLAM. The inverse risk matrix is then $\Omega_t^{-1} = \text{diag}(1/\sigma_i^2) = \text{diag}(p_i)$, where p_i is the *precision* (inverse variance) of asset i .

In this case, the full FLPE simplifies dramatically and, as shown in Appendix D.1 - equation (97), it reduces to

$$ER = \sqrt{N\bar{\mathbf{p}}}\text{Std}_P(\mathbf{d}_t) \quad (25)$$

where $\bar{\mathbf{p}}$ is the average (return) precision of the securities and $\text{Std}_P(\mathbf{d}_t)$ is the precision weighted standard deviation of the enhancement signal (i.e, a measure of the cross-sectional dispersion of the known characteristics).

For our carbon example, (25) specializes to

$$\begin{array}{l} \text{Portfolio} \\ \text{Enhancement} \\ \text{i.e. } CO_2 \text{ abatement} \end{array} \approx \begin{array}{c} \text{Tracking} \\ \text{Error} \end{array} * \sqrt{\begin{array}{cc} \text{Breadth} * & \text{Average Precision} * \text{Weighted Variance} \\ & \text{of Enhancement Signal} \end{array}} \quad (26)$$

This formulation can be used to explain why the enhancement to tracking error trade off is different in different markets; or in the same market why the trade off is different for different objectives. It also reiterates that, as discussed previously, if all stocks had the same carbon intensity then no matter what active weights the manager chose, they couldn't lower the aggregate intensity of the portfolio. Conversely, the larger the dispersion, the easier it is to lower the portfolio intensity.

6 Empirical Analysis

Our empirical analysis has two main objectives. Our first objective (section 6.2.1) is to use the FLPE to generate trade-off curves between improvements in the carbon intensity (CO₂) of large cap equity portfolios and the level of portfolio tracking error (ex-ante and ex-post). The second objective (in section 6.2.2), is to quantify the components of the maximised squared ER (the active, static and cross terms in equation 13) for the same portfolios. This analysis will also produce curves between the maximised squared ER and percentage CO₂ improvements.

The analysis is performed using equity portfolios for three developed economies - Australia, USA and the UK. We select these three countries as we expect different trade off curves because they have large cap equities in very different sectors. Australia is predominantly in the Financials and Materials (mining and resources) sectors, with other significant industries including Health, Industrials, and Consumer Staples. The USA is concentrated in the Technology industry, with companies like Nvidia, Apple, and Microsoft dominating by market capitalization. Other sectors with significant large-cap companies include Healthcare, Financial Services, and Energy. The UK's large cap stocks represent a diverse mix of industries. The most prominent sectors include Finance, Pharmaceuticals, Energy (Oil & Gas), and Mining, however it also includes industrials, consumer goods, telecommunications and hospitality.

6.1 Data

Returns and market capitalization for individual stocks are sourced from Refinitiv for Australia and the UK and from CRSP for the US. Earnings per share, dividend yield and price to book ratios are obtained from Refinitiv for all three countries.

We use Refinitiv to also obtain the carbon intensity for each stock defined as the sum of scope 1 and 2 emissions in tonnes per one million in USD denominated revenue. Refinitiv reports

a broad coverage of the scope 1 and 2 emissions by fiscal year (FY) from 2013. Historically, the time for firms to report CO2 emissions has varied widely, with a typical data lag of one to three years due to voluntary reporting and data collection challenges. Under new mandatory reporting regimes (such as in Australia), companies are now required to report within three to four months after the financial year-end. However this requirement is not in effect until 2025. We also acknowledge this is further complicated by firms that have year ends in March, June, September and December. We therefore use the reported FY CO2 intensity data with a considerable time lag. To illustrate, we consider a portfolio allocation in 2015. Over January 2015 to June 2015 we use the CO2 data from the 2013 FY. For the July to December allocations, we use CO2 for the 2014 FY.

Our out of sample period starts in January 2015 and ends December 2024 - 120 monthly observations. Our portfolios are rebalanced each month conditional on the information set. This means returns and factors (discussed below) are updated monthly, but emissions are updated bi-annually (at the beginning of January and July of each year).

For each market we initially consider a portfolio of stocks consisting of the top 100 stocks by market cap. Over the earlier part of the sample, CO2 data was often unavailable, meaning that only 70-80% of the stocks (70-80 stocks) could be considered. This percentage increased when we only considered the larger cap stocks (typically 80-95%). Our initial investable universe was therefore the top 50 stocks by market capitalisation for the USA and Australia and the top 30 stocks for the UK.

6.2 Methodology

The analysis is designed to replicate portfolios optimized in real-time. To do so, we evaluate the out of sample portfolio performance using 5 year rolling windows. This window length is long enough to provide reasonable parameter estimates and unconditional estimates, but it is also short enough to be responsive to structural breaks. This latter feature is potentially

important given that our out of sample period includes COVID-19. It is also consistent with the window size used by Ding and Martin (2017).

At the beginning of each year, the investable universe is constructed based on the then-current index constituents. This accounts for changes in index composition over time and replicates the information set available to a portfolio manager in real time.

To optimise portfolios we consider varying degrees of risk aversion $\lambda = 3, 5, 7$ and set $a = 1$. Given that $a, b \geq 0$, increases in b should enhance the portfolio by increasing the relative importance of the deterministic component (d_t). If d_t represents CO2 intensity, higher levels of b will tilt our portfolios to high CO2 intensity firms. Our objective is to reduce CO2 intensity, so we set d_t equal to the normalised inverse of carbon intensity $(x - \min)/(\max - \min)$ where x is the inverse of CO2 intensity and max and min are the cross-sectional maximum and minimum. This ensures that low CO2 intensity firms will increase enhancement more than high CO2 intensity firms. The normalisation ensures the inverse of CO2 intensity is bounded between zero and one. This helps to ensure the optimal solution is unaffected by a) CO2 outliers b) the difference in scale between standardised stock returns (defined below) and CO2 intensity and c) the significant positive skew in the cross-section of CO2 intensity data across firms. To achieve adequate levels of CO2 reduction we consider enhancement objectives with $b = [0, 0.50, 0.75, 1, 1.25, 1.50, 1.75, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]$.

We consider the analysis from an Australian investors perspective. To remove the influence of exchange rates when comparing the results between the Australian, US and UK markets, we implement foreign currency hedging. To illustrate, for an asset originally denominated in USD, the hedged return in AUD is given by:

$$1 + r_{AUD} = \frac{1 + r_{f,AUD}}{1 + r_{f,USD}} + (1 + r_S)r_{USD} \quad (27)$$

where r_{AUD} is the US hedged return for the asset in AUD, r_{USD} is the continuously compounded return of the asset in USD ($\ln(P_t) - \ln(P_{t-1})$), $r_{f,USD}$ is risk-free rate in USD, $r_{f,AUD}$

the risk-free rate in AUD and r_s the spot currency return defined as $FX_t/FX_{t-1} - 1$ where FX_t is the number of USD to purchase 1 AUD at time t . All asset returns are scaled by 100.

In the exposition that follows, to assist the reader and improve the flow of the paper, we restate some of the above equations. For each 5 year estimation window over the out of sample period i.e conditional on the monthly information set at time $t-1$ (December 2014 to November 2024), we perform the following

Step 1: Create standardised residual returns ($\tilde{r}_{i,t}$)

We run the following OLS regression for each stock

$$r_{i,t-1}^{Tot} = A + B(r_{m,t-1}) + r_{i,t-1} \quad (28)$$

where $r_{i,t-1}^{Tot}$ is the total continuously compounded return for stock i less the risk free rate proxied by the 3 month T-bill rate, $r_{m,t-1}$ is the market return proxied by the S&P500, ASX200, or FTSE100 and $r_{i,t-1}$ is the residual return. To create the standardised residual returns

$$\tilde{r}_{i,t-1} = r_{i,t-1} / \sigma_{i,t-1} \quad (29)$$

we employ the following GARCH(1,1) process

$$\sigma_{i,t-1}^2 = \sigma_i^2(1 - \alpha - \beta) + \alpha r_{i,t-2}^2 + \beta \sigma_{i,t-2}^2 \quad (30)$$

Following Ding and Martin (2017) we set $\alpha = 0.1$ and $\beta = 0.81$, and set σ_i^2 equal to the average of $r_{i,t}^2$. To initialise estimates $r_{i,0}^2 = \sigma_{i,0}^2 = \sigma_i^2$

Step 2: Create the unconditional estimate of the factor returns (IC)

To estimate the *unconditional* IC at time $t-1$, we first estimate ICs or factor returns conditional on the information set at time $t-1$ (IC_{t-1}). To do so, we estimate the following

cross-sectional regression

$$\tilde{r}_{t-1} = Z_{t-2}IC_{t-1} + \epsilon_{t-1} \quad (31)$$

where $\tilde{r}_{t-1}$ is a $N \times 1$ vector of standardised returns created via step 1, Z_{t-2} is a $N \times K$ matrix of standardised factors (discussed below), IC_{t-1} is the estimated $K \times 1$ slope coefficient of stochastic factors, and ϵ_{t-1} is a $N \times 1$ vector of residuals.

We follow Ding and Martin (2017) and employ four standardised factors ($K = 4$): earnings per share, dividend yield, price to book and momentum defined as the cumulative 12-month return from t-13 to t-2. We calculate the standardised factors via

$$z_{i,k,t-2} = \frac{x_{i,k,t-2} - \mu_{i,k,t-2}}{\sigma_{i,k,t-2}} \quad (32)$$

where $x_{i,k,t-2}$ is the k th factor for stock i , $\mu_{i,k,t-2}$ is the cross-sectional mean for the k th factor at time t-2, i.e $\mu_{i,k,t-2} = \bar{x}_{k,t-2} = \sum_{i=1}^N x_{i,k,t-2}$ and $\sigma_{i,k,t-2} = \sqrt{0.1(x_{i,k,t-3} - \bar{x}_{k,t-3})^2 + 0.9\sigma_{i,k,t-3}^2}$ which is a factor and stock-specific exponentially weighted moving average initialised via $\sigma_{i,k,0}^2 = (x_{i,k,0} - \bar{x}_{k,0})^2 = \frac{\sum_{p=3}^{62} (x_{i,k,t-p} - \bar{x}_{k,t-p})^2}{60}$.

Given our 5 year rolling window, we repeat the above cross-sectional regression each month going back five years i.e from t-60 to t-1. To calculate the $K \times 1$ *unconditional* IC vector, we take the time series average of the conditional IC_t vectors i.e

$$IC = \sum_{p=1}^{60} IC_{t-p}/60 \quad (33)$$

Step 3: Calculate the weight vector at time t

For a given level of risk aversion (λ), conditional on the information set at time t-1, we solve the portfolio weights for the next month (time t) for $a = 1$ and each of the values for b . The

solution to the unconstrained optimisation problem is

$$\Delta w_t = \frac{1}{2\lambda} \Omega_{c,t}^{-1} u_{obj,t} \quad (34)$$

where $\Delta w_t = w_t - w_{b,t}$, where w_t is the $N \times 1$ vector of weights and $w_{b,t}$ is the $N \times 1$ vector of benchmark weights, $\Omega_{c,t}^{-1}$ is an $N \times N$ matrix equal to $\Omega_t^{-1} - \frac{\Omega_t^{-1} 1 1' \Omega_t^{-1}}{1' \Omega_t^{-1} 1}$ and $\mu_{obj,t} = E[e_t | I_{t-1}] = a\alpha_t + b d_t$ where $\alpha_t = \wedge_t^{\frac{1}{2}} Z_{t-1}' IC$, $\wedge_t^{\frac{1}{2}}$ is a diagonal $N \times N$ matrix constructed via the GARCH(1,1) estimates conditional on the information at time t-1 (step 1), Z_{t-1} is observable at t-1 and IC is per equation 33.

The risk matrix (Ω_t) is an $N \times N$ matrix calculated via

$$\Omega_t = \wedge_t^{\frac{1}{2}} (Z_{t-1} \Sigma_{IC} Z_{t-1}' + \sigma_\epsilon^2 I) \wedge_t^{\frac{1}{2}} \quad (35)$$

where Σ_{IC} is the $K \times K$ unconditional covariance matrix of ICs (factor returns) calculated using the time series of ICs up to $t - 1$ (step 2), I is an $N \times N$ identity matrix, and σ_ϵ^2 is the *unconditional* idiosyncratic error variance calculated via

$$\sigma_\epsilon^2 = (1 - \sum_{k=1}^K (\sigma_{IC,k}^2 + IC_k^2)) \quad (36)$$

where $\sigma_{IC,k}^2$ is the k^{th} diagonal element of Σ_{IC} and IC_k is the k^{th} element of IC (equation 33).

Given that portfolio managers are subject to constraints we also consider a numerically optimised solution to the following

$$Max \ U_t = \Delta w_t' \mu_{obj,t} - \lambda (\Delta w_t' \Omega_t \Delta w_t) \quad (37)$$

Subject to

$$1' \Delta w_t = 0 \quad (38)$$

$$0 \leq w_{i,t} \quad (39)$$

$$-\psi < \Delta w_t < \psi \quad (40)$$

where U_t is the investor's utility, and ψ is the minimum and maximum deviation from the benchmark weight (box constraint). We consider values of 0.005, 0.01, 0.025 and 0.05. The weight vector at time t conditional on the information at time $t - 1$ (w_t) enables calculation of the realised portfolio return for each value of b for month t .

We now note the impact of missing observations on the above analysis. A stock is only included in the optimization at $t - 1$ if all of the following conditions hold a) CO2 data is available (as defined above); b) five years of historical monthly stock return data is available (to estimate equation 28) and standardize via equations 29) and 30) (the GARCH specification requires a burn in period); c) all monthly factors are available over an historical five year period (required to compute the exponentially weighted moving average for equation 32). This means that the number of stocks included in the optimization (N) varies through time and is less than or equal to N^* the total number of stocks ($N < N^*$) where $N^* = 50$ (Aust and USA) and $N^* = 30$ (UK).

Given the missing data, we construct a synthetic benchmark (weights and returns). At the beginning of the each year the benchmark weights are equal to market capitalization weights (where only the N stocks remaining in the portfolio are included). For the remaining months of the year, we update the market capitalization and hence the benchmark weights using the realized stock returns.

The analytical solution for Δw_t sums to zero. We therefore apply the change in weights from the benchmark to the stocks included in the optimization (N).³

³When using the four factors, the UK data set could not generate well defined positive definite estimates of the risk matrix (Ω_t) across large parts of the OOS period (at least 50% of the time). This is because the idiosyncratic error variance σ_ϵ^2 (equation 36) was negative. This was due to the high variance in the dividend yield (primarily) and book to market factors. Despite their standardization via the EWMA, the estimated ICs (via equation 33) were highly volatile resulting in $\sum_{k=1}^K (\sigma_{IC,k}^2 + IC_k^2) < 1$. Via equation 35 this meant the diagonal elements of Ω_t were negative. In response we allowed the EWMA to update

Step 4: Calculate ex-ante tracking error and maximised squared ER

For each level of risk aversion (λ) and enhancement (b), we generate real time out of sample plots. To do so we calculate the ex-ante tracking error (EA TE) for each month via

$$EA\ TE = (\Delta w_t' \Omega_t \Delta w_t)^{0.5} \quad (41)$$

We also use our forecasts to calculate the active, static and cross-term components of the unconstrained maximized squared ER via equation 13.

At the end of the OOS period, we calculate the ex-post tracking error (EP TE) via

$$EP\ TE = \sigma_{(r_{p,t} - r_{b,t})} = \sigma_{\Delta w' r_t} \quad (42)$$

where $r_{p,t}$ denotes the portfolio return, $r_{b,t}$ the monthly benchmark return and r_t the $N \times 1$ vector of monthly realized returns.

For each of the values for b , we also use the optimized weight vectors over the OOS period to calculate the average CO2 intensity for the portfolio. We then calculate the percentage improvement in CO2 intensity relative to the average CO2 intensity for the portfolio when $b = 0$.

For each market c , constraint set s , and enhancement level b , we evaluate whether the model-implied (ex-ante) tracking error aligns with realized (ex-post) tracking error computed from the 120 monthly active returns. Specifically, we test the null hypothesis that the true monthly variance equals the ex-ante variance implied by the optimizer. Under the standard

more quickly in response to new information (changed the parameters from 0.1 and 0.9 to 0.3 and 0.7), and removed stocks with negative book to market valuations (significant culprits were Rolls Royce, BT Group and GlaxoSmithKline). Both measures helped a little, but the diagonal elements were still < 0 . For the UK dataset, we therefore removed the dividend yield factor completely and the negative book to market observations. This resulted in positive idiosyncratic variances and well defined risk matrices.

i.i.d. normal approximation for monthly active returns, the statistic

$$Q_{c,s,b} = (T - 1) \frac{(EPTE_{c,s,b})^2}{(EA TE_{c,s,b})^2} \quad (43)$$

has an asymptotic χ^2_{T-1} distribution, with $T = 120$.

6.3 Results

Tables 1 to 3 report summary statistics for firm specific CO2 intensity by year from 2013 to 2023 (the data used to optimize the portfolios over the OOS period). The CO2 intensity is the sum of scope 1 and 2 CO2 equivalent emissions in tonnes per 1M USD in revenue. On average Australian stocks are more carbon intensive (364t/1M USD revenue) than US (49t) and UK stocks (157t). Greater levels of cross-sectional variation in CO2 should mean lower levels of tracking error for a given percentage decrease in portfolio CO2. The UK has the lowest coefficient of variation (CV) at 1.62, so our prior is that the costs of CO2 reduction (TE) will be higher for the UK than Australia and the USA.

Table 1: ASX: Scope 1 and 2 CO2t per 1M USD in revenue

Year	mean	median	min	max	std dev	CV
2013	274.1	52.7	1.2	2315.2	505.2	1.843
2014	244.3	56.4	1.3	2023.2	432.5	1.770
2015	507.2	67.1	1.6	6346.6	1213.9	2.393
2016	574.3	70.9	2.3	5437.0	1217.1	2.119
2017	524.8	65.3	2.1	4542.1	1049.6	2.000
2018	324.8	50.4	0.4	3974.0	734.4	2.261
2019	318.4	49.6	0.9	4138.0	752.3	2.363
2020	407.9	48.2	1.1	6267.5	1025.6	2.514
2021	298.8	25.5	0.8	5606.3	866.6	2.900
2022	244.9	22.8	0.7	3322.9	576.5	2.354
2023	287.4	19.3	0.2	5313.5	826.4	2.876
Average	364.3	48.0	1.1	4480.6	836.4	2.309

Reports CO2 intensity descriptive statistics for the data used to optimise portfolios over the OOS period. CV denotes the coefficient of variation.

Table 2: USA: Scope 1 and 2 CO₂t per 1M USD in revenue

Year	mean	median	min	max	std dev	CV
2013	54.0	32.0	2.0	340.5	76.5	1.417
2014	52.6	34.0	2.0	360.1	72.9	1.386
2015	55.8	25.1	1.3	524.5	107.2	1.921
2016	55.3	21.5	1.0	615.5	121.7	2.200
2017	50.4	21.4	0.9	512.0	99.9	1.982
2018	51.0	21.0	0.7	440.9	90.5	1.774
2019	50.1	26.2	0.7	456.6	90.5	1.808
2020	52.8	22.4	0.7	603.4	116.2	2.202
2021	40.6	17.8	1.8	379.2	75.4	1.856
2022	35.4	16.3	1.6	260.9	53.7	1.514
2023	38.4	16.7	1.5	298.8	62.5	1.629
Average	48.8	23.1	1.3	435.7	87.9	1.790

Reports CO₂ intensity descriptive statistics for the data used to optimise portfolios over the OOS period. CV denotes the coefficient of variation.

Table 3: UK: Scope 1 and 2 CO₂t per 1M USD in revenue

Year	mean	median	min	max	std dev	CV
2013	161.5	29.6	1.5	867.1	239.9	1.486
2014	153.0	32.2	1.3	792.8	227.8	1.489
2015	219.7	39.5	1.8	918.8	305.2	1.389
2016	215.4	35.1	1.1	1043.7	304.8	1.415
2017	189.4	34.7	0.8	1085.7	277.8	1.466
2018	163.2	34.6	2.6	1427.4	296.6	1.817
2019	182.7	24.8	0.4	1295.3	316.4	1.732
2020	181.2	18.5	1.3	1268.7	323.9	1.787
2021	100.5	20.0	0.1	812.2	180.7	1.799
2022	83.0	20.6	0.1	552.0	145.3	1.751
2023	79.1	17.0	0.1	483.0	135.2	1.710
Average	157.1	27.9	1.0	958.8	250.3	1.622

Reports CO₂ intensity descriptive statistics for the data used to optimise portfolios over the OOS period. CV denotes the coefficient of variation.

Figure 1: Number of stocks in the portfolio (N) over the OOS period

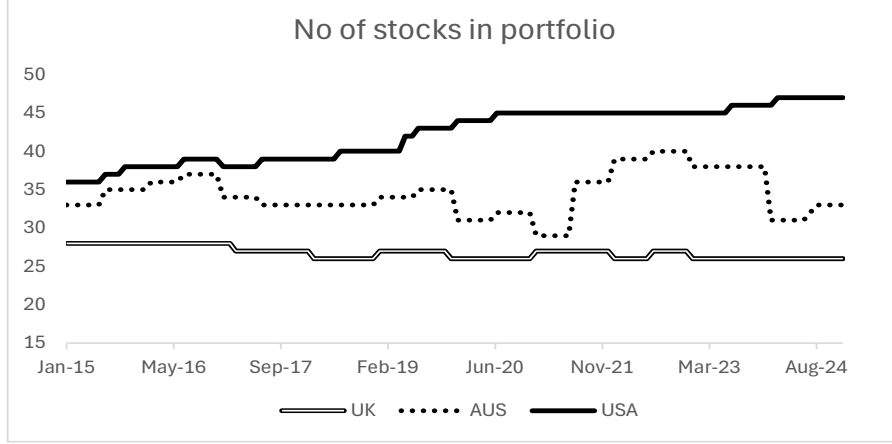


Figure 1 displays the number of stocks included in the portfolio optimization through time (N). The upward trajectory for the US is largely due to a higher number of firms reporting their CO2 intensity. For most firms in the UK, CO2 data is available over the entire period, so the slight downward trajectory for the UK is due to missing return or factor data.

6.3.1 Tracking error versus percentage decrease in CO2 intensity

Tables 4 to 6 present performance metrics over the OOS period from January 2015 to December 2024 when $\lambda = 3$. CO2(%) is the percentage decrease in CO2 of the portfolio relative to the optimized portfolio with $b = 0$ (no enhancement). EP TE is the ex post percentage tracking error relative to the benchmark. EA TE is the ex ante tracking error. We report results for the unconstrained portfolio as well as constrained portfolios with box constraints of $\psi = 0.005, 0.01, 0.025, 0.05$.

As expected, the ex ante TE is consistently less than the ex post TE. Tighter box constraints (via decreases in ψ) reduce tracking error, but make it harder to reduce CO2. To illustrate, for Australia, when $b = 6$ the unconstrained portfolio achieves a 21.52% decrease in CO2 with an ex post (ex ante) TE of 0.442% (0.259%). When $\psi = 0.005$ the portfolio can only achieve a 9.325% reduction in CO2, however the ex post (ex ante) TE is reduced to 0.139% (0.084%).

We formally test for differences between ex-ante and ex-post TE using the χ^2 statistic in equation (43). We find that for modest levels of enhancement - namely b values up to 1 (UK) or 2 (US)- we cannot reject the null that the EP TE equals the EA TE. However, the two appear to be statistically different for higher values of b . In Australia, the null is rejected even at very low levels of enhancement. These inferences do not vary in any meaningful way as box constraints get tighter.

Figures 2 to 4 plot the ex ante and ex post TE(%) for the unconstrained and constrained portfolios against the percentage decrease in CO2 when $\lambda = 3$ for Australia, the USA and UK respectively. The figures clearly display the trade-offs involved. In order to further evaluate the impact of constraints, the last plot for each figure considers values of b up to 40. To illustrate, this enables the CO2 reduction for Australia to increase from 19% ($b=13$) to 28% ($b=40$).

Figure 5 uses the same data to enable cross country comparisons. We plot the unconstrained portfolios as well as the constrained portfolios with box constraints of $\psi = 0.01$. The UK shows higher levels of tracking error for a given percentage reduction in CO2. This is consistent with the UK having the lowest level of cross-sectional variation in CO2 over the sample period (measured by the coefficient of variation).

The constrained and unconstrained allocations are stable, and the unconstrained portfolios rarely result in negative weights (details available on request). For illustrative purposes, Tables 7 to 9 report turnover for portfolios with box constraints of $\psi = 0.01$, where turnover is calculated via

$$\sum_{i=1}^N |w_{i,t} - w_{i,t-1}| \quad (44)$$

This is reported across the entire OOS period (120 months) as well as across all months excluding the reallocation that occurs at the beginning of each year (108 months). This is because most of the turnover at the beginning of the year is due to changes in market capitalization and stocks entering/exiting the top 50/30.

Table 4: Australia: Tracking error against CO2 percentage decrease ($\lambda = 3$)

b	Unconstrained				$\psi = 0.005$				$\psi = 0.01$				$\psi = 0.025$				$\psi = 0.05$			
	CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)	
0	0.000	0.037	0.038		0.000	0.037	0.037		0.000	0.037	0.038		0.000	0.037	0.038		0.000	0.037	0.038	
0.5	1.794	0.048	0.044*		1.721	0.046	0.043*		1.784	0.047	0.044*		1.793	0.048	0.044*		1.793	0.048	0.044*	
0.75	2.690	0.061	0.050***		2.494	0.055	0.047***		2.636	0.061	0.049***		2.689	0.061	0.050***		2.688	0.061	0.050***	
1	3.587	0.077	0.058***		3.185	0.064	0.052***		3.467	0.073	0.056***		3.586	0.077	0.058***		3.587	0.077	0.058***	
1.25	4.484	0.094	0.066***		3.799	0.072	0.056***		4.276	0.086	0.062***		4.477	0.094	0.066***		4.483	0.094	0.066***	
1.5	5.381	0.111	0.075***		4.342	0.079	0.059***		5.057	0.098	0.068***		5.349	0.110	0.075***		5.381	0.111	0.075***	
1.75	6.277	0.129	0.085***		4.832	0.085	0.062***		5.788	0.109	0.074***		6.204	0.127	0.083***		6.278	0.129	0.085***	
2	7.174	0.147	0.094***		5.279	0.090	0.064***		6.466	0.119	0.080***		7.047	0.143	0.091***		7.175	0.147	0.094***	
3	10.761	0.220	0.134***		6.669	0.108	0.071***		8.750	0.153	0.098***		10.344	0.200	0.122***		10.731	0.217	0.133***	
4	14.348	0.293	0.175***		7.684	0.119	0.075***		10.597	0.179	0.111***		13.372	0.250	0.152***		14.006	0.279	0.168***	
5	17.935	0.367	0.217***		8.567	0.129	0.081***		12.122	0.200	0.120***		15.986	0.294	0.178***		16.973	0.333	0.200***	
6	21.522	0.442	0.259***		9.325	0.139	0.084***		13.399	0.217	0.126***		18.212	0.329	0.201***		19.680	0.381	0.230***	
7	25.109	0.516	0.301***		9.958	0.146	0.089***		14.512	0.229	0.132***		20.170	0.359	0.219***		22.196	0.429	0.260***	
8	28.696	0.590	0.344***		10.368	0.148	0.089***		15.488	0.240	0.137***		21.937	0.386	0.235***		24.557	0.473	0.288***	
9	32.283	0.664	0.386***		10.790	0.152	0.092***		16.390	0.250	0.142***		23.512	0.411	0.248***		26.886	0.514	0.316***	
10	35.870	0.739	0.429***		11.197	0.160	0.095***		17.077	0.258	0.146***		24.952	0.433	0.259***		29.007	0.552	0.342***	
11	39.457	0.813	0.471***		11.656	0.161	0.097***		17.771	0.266	0.149***		26.298	0.453	0.269***		30.963	0.585	0.365***	
12	43.044	0.887	0.513***		11.929	0.163	0.095***		18.425	0.273	0.153***		27.546	0.472	0.278***		32.738	0.614	0.386***	
13	46.631	0.962	0.556***		12.191	0.166	0.098***		19.022	0.279	0.156***		28.731	0.490	0.286***		34.439	0.643	0.405***	

Reports ex post (EP) and ex ante (EA) tracking error % against the percentage decrease in CO2 for varying levels of enhancement (b) and box constraints ψ . Results are for the entire OOS period from January 2015 to December 2024 with monthly re-balancing. *, **, *** denotes a statistically significant difference between the EP and EA TE at the 10%, 5% and 1% levels respectively.

Table 5: USA: Tracking error against CO2 percentage decrease ($\lambda = 3$)

b	Unconstrained				$\psi = 0.005$				$\psi = 0.01$				$\psi = 0.025$				$\psi = 0.05$			
	CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)	
0	0.000	0.037	0.040		0.000	0.037	0.040		0.000	0.037	0.040		0.000	0.037	0.040		0.000	0.037	0.040	
0.5	1.184	0.041	0.045		1.181	0.041	0.045		1.184	0.041	0.045		1.185	0.041	0.045		1.183	0.041	0.045	
0.75	1.776	0.048	0.050		1.748	0.046	0.049		1.777	0.048	0.050		1.778	0.048	0.050		1.774	0.048	0.050	
1	2.368	0.057	0.056		2.247	0.050	0.052		2.367	0.056	0.056		2.368	0.057	0.056		2.367	0.057	0.056	
1.25	2.960	0.067	0.064		2.689	0.054	0.055		2.940	0.064	0.063		2.960	0.067	0.064		2.959	0.067	0.064	
1.5	3.552	0.077	0.072		3.111	0.058	0.057		3.481	0.071	0.069		3.553	0.077	0.072		3.551	0.077	0.072	
1.75	4.144	0.089	0.080*		3.516	0.061	0.059		3.997	0.077	0.073		4.144	0.089	0.080*		4.143	0.089	0.080*	
2	4.736	0.100	0.089**		3.905	0.065	0.061		4.487	0.083	0.078		4.736	0.100	0.089**		4.734	0.100	0.089**	
3	7.104	0.148	0.126***		5.364	0.081	0.069***		6.261	0.103	0.090***		7.034	0.139	0.122**		7.102	0.148	0.126***	
4	9.472	0.196	0.165***		6.660	0.095	0.077***		7.917	0.122	0.101***		9.180	0.170	0.148**		9.463	0.196	0.165***	
5	11.840	0.245	0.204***		7.760	0.106	0.083***		9.474	0.142	0.111***		11.167	0.196	0.168***		11.753	0.240	0.202***	
6	14.208	0.295	0.243***		8.924	0.123	0.091***		10.942	0.159	0.120***		12.956	0.217	0.183***		13.898	0.275	0.234***	
7	16.575	0.344	0.283***		9.712	0.126	0.093***		12.303	0.175	0.129***		14.624	0.239	0.197***		15.952	0.308	0.261***	
8	18.943	0.394	0.323***		10.450	0.132	0.097***		13.550	0.189	0.137***		16.201	0.261	0.209***		17.913	0.339	0.286***	
9	21.311	0.444	0.363***		11.022	0.139	0.102***		14.729	0.203	0.146***		17.723	0.282	0.221***		19.757	0.366	0.307***	
10	23.679	0.493	0.403***		11.583	0.144	0.105***		15.803	0.216	0.152***		19.199	0.302	0.233***		21.497	0.390	0.325***	
11	26.047	0.543	0.443***		12.097	0.151	0.110***		16.833	0.227	0.159***		20.631	0.323	0.244***		23.134	0.413	0.341***	
12	28.415	0.593	0.483***		12.389	0.152	0.113***		17.813	0.238	0.166***		22.030	0.342	0.254***		24.680	0.435	0.356***	
13	30.783	0.643	0.523***		12.667	0.159	0.115***		18.733	0.247	0.171***		23.393	0.361	0.264***		26.166	0.455	0.369***	

Reports ex post (EP) and ex ante (EA) tracking error % against the percentage decrease in CO2 for varying levels of enhancement (b) and box constraints ψ . Results are for the entire OOS period from January 2015 to December 2024 with monthly re-balancing. *, **, *** denotes a statistically significant difference between the EP and EA TE at the 10%, 5% and 1% levels respectively.

Table 6: UK: Tracking error against CO2 percentage decrease ($\lambda = 3$)

b	Unconstrained				$\psi = 0.005$				$\psi = 0.01$				$\psi = 0.025$				$\psi = 0.05$			
	CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)		CO2 %	EP(%)	EA(%)	
0	0.000	0.041	0.034***		0.000	0.041	0.034***		0.000	0.041	0.034***		0.000	0.041	0.034***		0.000	0.039	0.034**	
0.5	0.877	0.041	0.040		0.855	0.041	0.040		0.880	0.041	0.040		0.860	0.042	0.040		0.793	0.041	0.040	
0.75	1.315	0.048	0.046		1.209	0.046	0.043		1.312	0.048	0.045		1.302	0.049	0.046		1.222	0.050	0.045**	
1	1.753	0.058	0.052**		1.516	0.052	0.046**		1.717	0.057	0.051**		1.745	0.058	0.052**		1.639	0.060	0.051***	
1.25	2.191	0.070	0.059***		1.786	0.056	0.049***		2.088	0.066	0.056***		2.140	0.070	0.059***		2.041	0.070	0.058***	
1.5	2.630	0.082	0.066***		2.035	0.061	0.051***		2.439	0.075	0.061***		2.610	0.082	0.066***		2.533	0.080	0.066***	
1.75	3.068	0.095	0.074***		2.268	0.065	0.053***		2.759	0.083	0.065***		3.027	0.094	0.074***		2.846	0.101	0.073***	
2	3.506	0.108	0.082***		2.492	0.069	0.055***		3.058	0.090	0.069***		3.465	0.107	0.081***		3.305	0.113	0.081***	
3	5.259	0.163	0.115***		3.251	0.081	0.060***		4.106	0.112	0.080***		5.009	0.152	0.108***		4.963	0.160	0.114***	
4	7.012	0.219	0.149***		3.895	0.091	0.065***		5.016	0.130	0.088***		6.359	0.192	0.130***		6.559	0.215	0.148***	
5	8.766	0.275	0.184***		4.432	0.099	0.070***		5.811	0.145	0.096***		7.618	0.224	0.147***		8.273	0.274	0.178***	
6	10.519	0.332	0.219***		4.895	0.106	0.074***		6.550	0.157	0.102***		8.625	0.253	0.161***		9.567	0.304	0.204***	
7	12.272	0.389	0.254***		5.308	0.111	0.078***		7.243	0.169	0.109***		9.650	0.272	0.173***		10.750	0.353	0.227***	
8	14.025	0.446	0.290***		5.641	0.115	0.080***		7.886	0.179	0.114***		10.637	0.292	0.183***		12.001	0.388	0.249***	
9	15.778	0.503	0.325***		5.952	0.118	0.083***		8.518	0.186	0.121***		11.619	0.312	0.192***		13.049	0.421	0.268***	
10	17.531	0.560	0.361***		6.228	0.121	0.085***		8.989	0.194	0.125***		12.335	0.334	0.201***		14.242	0.455	0.286***	
11	19.284	0.617	0.396***		6.484	0.123	0.088***		9.470	0.201	0.129***		13.247	0.347	0.209***		15.413	0.480	0.300***	
12	21.037	0.674	0.432***		6.725	0.125	0.090***		9.918	0.208	0.134***		13.942	0.363	0.216***		16.462	0.505	0.315***	
13	22.790	0.731	0.468***		6.951	0.126	0.092***		10.357	0.217	0.139***		14.742	0.370	0.224***		17.481	0.525	0.329***	

Reports ex post (EP) and ex ante (EA) tracking error % against the percentage decrease in CO2 for varying levels of enhancement (b) and box constraints ψ . Results are for the entire OOS period from January 2015 to December 2024 with monthly re-balancing. *, **, *** denotes a statistically significant difference between the EP and EA TE at the 10%, 5% and 1% levels respectively.

Figure 2: Australia: Ex ante and ex post tracking error (%) versus CO2 reduction (%)

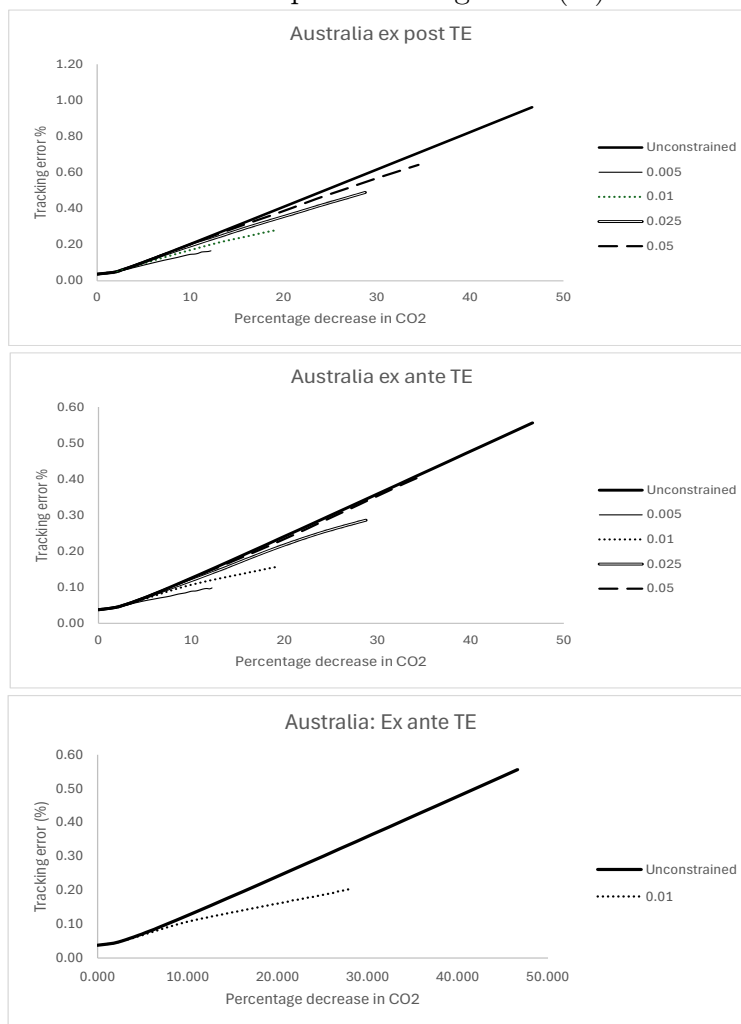


Figure 3: USA: Ex ante and ex post tracking error (%) versus CO2 reduction (%)

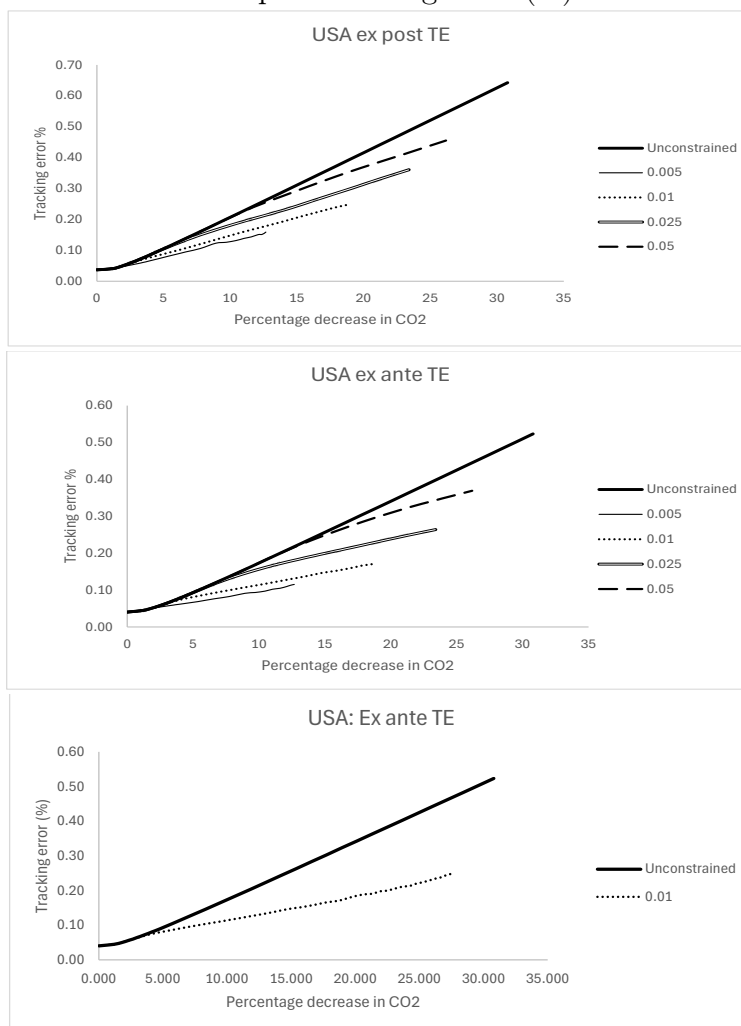


Figure 4: UK: Ex ante and ex post tracking error (%) versus CO2 reduction (%)

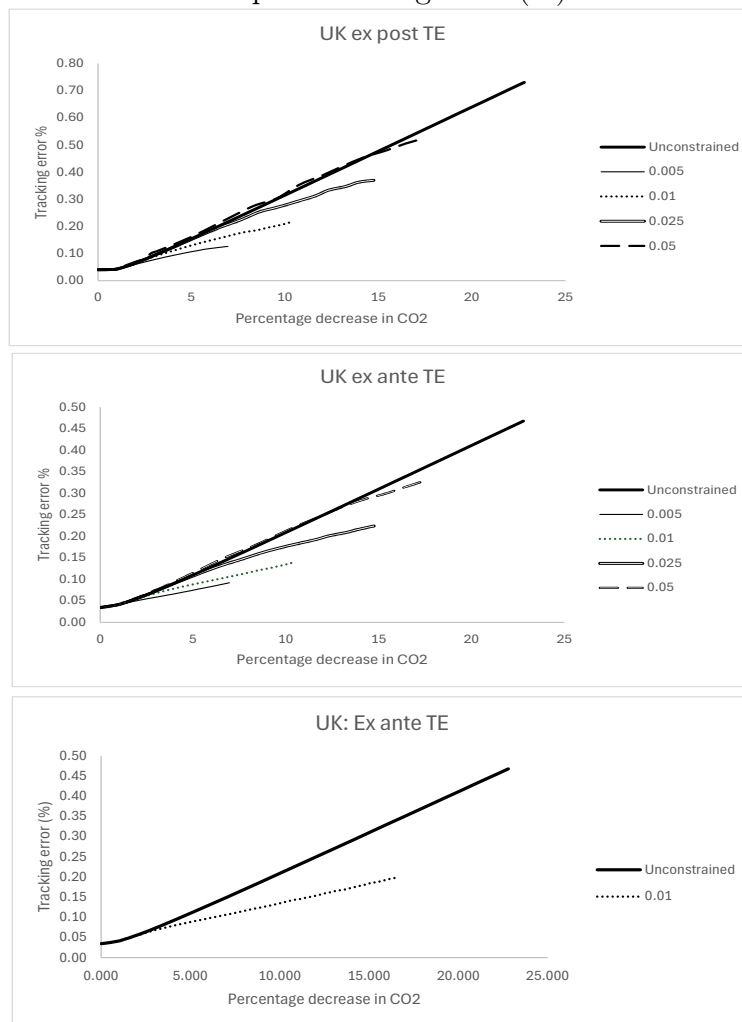
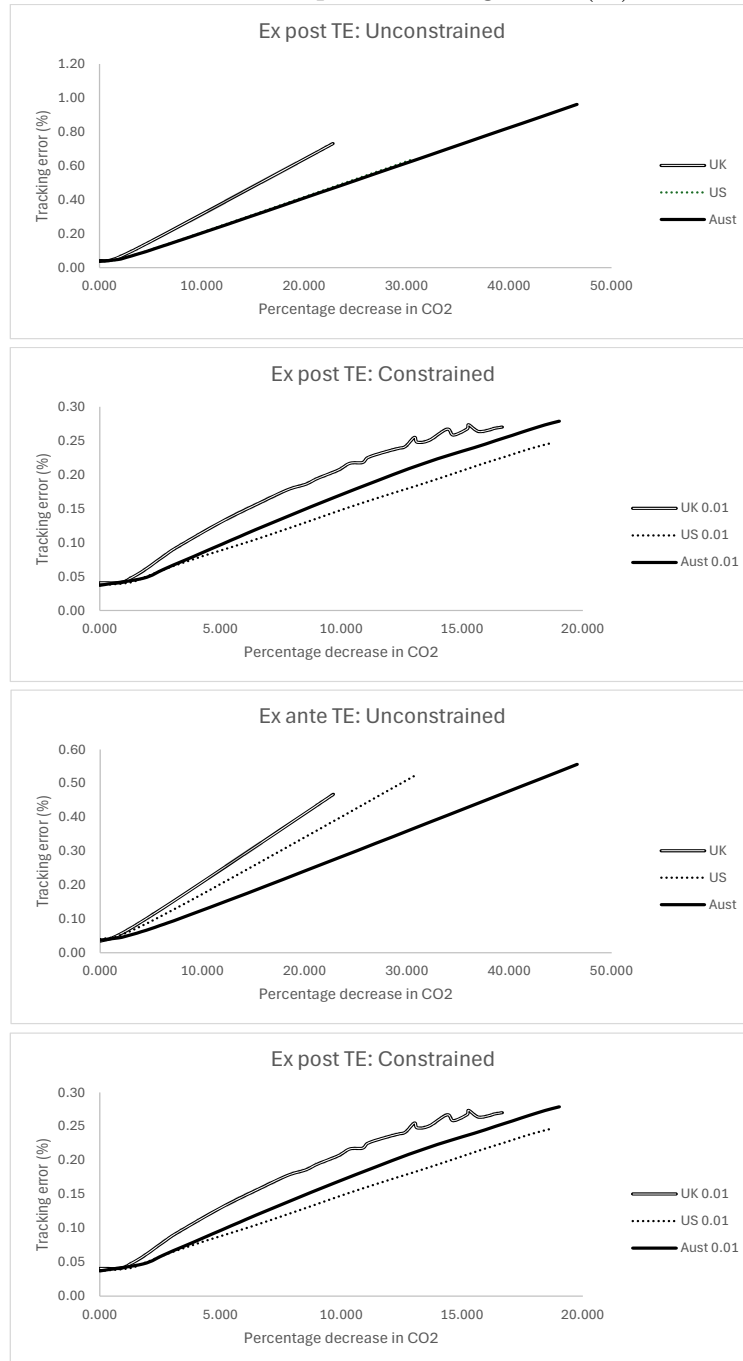


Figure 5: All countries: Ex ante and ex post tracking error (%) versus CO2 reduction (%)



The results reveal that higher levels of CO2 enhancement and lower levels of risk aversion slightly increase portfolio turnover. Even high levels of enhancement and low levels of risk aversion though produce very stable allocations.

We argue these stable allocations are due to the stability in the covariance matrix, which is in part explained by the standardization of the factor returns. We note that Ding and Martin (2017) promote the standardization of factors to enhance stability, but they did not apply their approach to a portfolio allocation problem. Our results therefore strongly support their methodology.⁴

Table 7: Australia: turnover by level of enhancement ($\psi = 0.01$)

	$\lambda = 3$		$\lambda = 5$		$\lambda = 7$	
b	All mths	no Jan	All mths	no Jan	All mths	no Jan
0	0.060	0.044	0.058	0.042	0.058	0.042
0.5	0.061	0.045	0.059	0.043	0.058	0.042
0.75	0.062	0.046	0.059	0.043	0.058	0.042
1	0.062	0.047	0.059	0.044	0.058	0.043
1.25	0.063	0.047	0.060	0.044	0.059	0.043
1.5	0.064	0.048	0.060	0.045	0.059	0.043
1.75	0.065	0.049	0.061	0.045	0.059	0.044
2	0.065	0.050	0.061	0.046	0.060	0.044
3	0.067	0.052	0.063	0.047	0.061	0.045
4	0.068	0.053	0.065	0.049	0.063	0.047
5	0.069	0.054	0.066	0.050	0.064	0.048
6	0.070	0.055	0.067	0.051	0.065	0.049
7	0.070	0.055	0.067	0.052	0.065	0.050
8	0.071	0.056	0.068	0.052	0.066	0.051
9	0.073	0.057	0.068	0.053	0.067	0.051
10	0.072	0.056	0.069	0.053	0.067	0.052
11	0.072	0.057	0.069	0.054	0.067	0.052
12	0.072	0.057	0.069	0.054	0.068	0.052
13	0.072	0.057	0.070	0.054	0.068	0.053

Reports portfolio turnover for varying degrees of enhancement (b) and risk aversion (λ). All mths denotes the turnover across the entire OOS period (January 2015 to December 2024), no Jan removes the month of January because most of the turnover is due to changes in market capitalisation and stocks entering/exiting the top 50.

⁴This is subject to the positive definiteness issues raised for the UK in the footnote above.

Table 8: USA: turnover by level of enhancement ($\psi = 0.01$)

b	$\lambda = 3$		$\lambda = 5$		$\lambda = 7$	
	All mths	no Jan	All mths	no Jan	All mths	no Jan
0	0.059	0.050	0.057	0.048	0.057	0.048
0.5	0.059	0.050	0.057	0.048	0.057	0.048
0.75	0.059	0.050	0.057	0.048	0.057	0.048
1	0.059	0.050	0.057	0.048	0.057	0.048
1.25	0.059	0.050	0.057	0.048	0.057	0.048
1.5	0.059	0.050	0.057	0.048	0.057	0.048
1.75	0.059	0.050	0.057	0.048	0.057	0.048
2	0.059	0.050	0.057	0.048	0.057	0.048
3	0.060	0.051	0.058	0.049	0.057	0.048
4	0.061	0.052	0.058	0.049	0.057	0.048
5	0.062	0.053	0.059	0.050	0.058	0.049
6	0.063	0.054	0.060	0.051	0.058	0.049
7	0.064	0.055	0.060	0.051	0.058	0.049
8	0.065	0.056	0.061	0.052	0.059	0.050
9	0.068	0.058	0.061	0.053	0.059	0.050
10	0.066	0.057	0.062	0.053	0.060	0.051
11	0.067	0.058	0.063	0.054	0.060	0.051
12	0.069	0.061	0.063	0.054	0.061	0.052
13	0.068	0.059	0.066	0.057	0.061	0.052

Reports portfolio turnover for varying degrees of enhancement (b) and risk aversion (λ). All mths denotes the turnover across the entire OOS period (January 2015 to December 2024), no Jan removes the month of January because most of the turnover is due to changes in market capitalisation and stocks entering/exiting the top 50.

Table 9: UK: turnover by level of enhancement ($\psi = 0.01$)

b	$\lambda = 3$		$\lambda = 5$		$\lambda = 7$	
	All mths	no Jan	All mths	no Jan	All mths	no Jan
0	0.064	0.044	0.063	0.044	0.063	0.044
0.5	0.064	0.045	0.063	0.044	0.063	0.044
0.75	0.064	0.045	0.063	0.044	0.063	0.044
1	0.065	0.045	0.064	0.044	0.063	0.044
1.25	0.065	0.046	0.064	0.045	0.063	0.044
1.5	0.065	0.046	0.064	0.045	0.064	0.044
1.75	0.065	0.046	0.064	0.045	0.064	0.044
2	0.065	0.046	0.064	0.045	0.064	0.044
3	0.066	0.047	0.065	0.046	0.064	0.045
4	0.066	0.047	0.065	0.046	0.065	0.046
5	0.067	0.048	0.065	0.046	0.065	0.046
6	0.067	0.048	0.066	0.047	0.065	0.046
7	0.068	0.049	0.066	0.047	0.065	0.046
8	0.068	0.049	0.067	0.047	0.066	0.047
9	0.069	0.050	0.067	0.048	0.066	0.047
10	0.069	0.050	0.067	0.048	0.066	0.047
11	0.069	0.050	0.067	0.048	0.066	0.047
12	0.069	0.050	0.068	0.049	0.067	0.048
13	0.070	0.051	0.068	0.049	0.067	0.048

Reports portfolio turnover for varying degrees of enhancement (b) and risk aversion (λ). All mths denotes the turnover across the entire OOS period (January 2015 to December 2024), no Jan removes the month of January because most of the turnover is due to changes in market capitalisation and stocks entering/exiting the top 30.

6.3.2 Total enhancement and its components via the FLPE

We now report the total enhancement results (maximum squared enhancement ratio (ER)) for varying degrees of enhancement as well as the active, static and cross term components (via equation 13) for the unconstrained optimisation. For illustrative purposes, the final column reports the squared generalised transfer coefficient (GTC) for the constrained portfolio with a box constraint of $\psi = 0.01$.

Tables 10 to 12 report the mean and the 5th and 95th percentiles of the total enhancement and each component across the OOS period (January 2015 to December 2024). Note the active component is invariant with respect to the level of enhancement, so it is reported in the caption for each table. Figure 6 plots the total maximum squared ER against the unconstrained percentage CO2 reduction for each country.

Table 10: Australia: Maximum squared ER and its components

b	CO2 %	Static			Cross			Total			GTC^2
		Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.066	0.009	0.195	1.000
0.5	1.794	0.024	0.005	0.045	0.006	-0.006	0.026	0.096	0.019	0.256	1.000
0.75	2.690	0.053	0.012	0.100	0.009	-0.010	0.038	0.128	0.024	0.320	1.000
1	3.587	0.095	0.022	0.179	0.012	-0.013	0.051	0.173	0.036	0.409	1.000
1.25	4.484	0.149	0.034	0.279	0.015	-0.016	0.064	0.229	0.049	0.520	1.000
1.5	5.381	0.214	0.048	0.402	0.018	-0.019	0.077	0.298	0.062	0.648	1.000
1.75	6.277	0.291	0.066	0.547	0.021	-0.022	0.089	0.378	0.076	0.793	1.000
2	7.174	0.380	0.086	0.714	0.024	-0.026	0.102	0.470	0.089	0.957	1.000
3	10.761	0.856	0.194	1.608	0.036	-0.038	0.153	0.958	0.188	1.807	0.999
4	14.348	1.521	0.345	2.858	0.048	-0.051	0.204	1.635	0.331	3.039	0.999
5	17.935	2.377	0.538	4.465	0.060	-0.064	0.255	2.503	0.518	4.599	0.997
6	21.522	3.422	0.775	6.430	0.073	-0.077	0.306	3.561	0.747	6.564	0.996
7	25.109	4.658	1.055	8.752	0.085	-0.089	0.357	4.808	1.019	8.895	0.993
8	28.696	6.084	1.378	11.431	0.097	-0.102	0.408	6.246	1.334	11.570	0.991
9	32.283	7.700	1.744	14.468	0.109	-0.115	0.459	7.875	1.693	14.592	0.987
10	35.870	9.506	2.154	17.861	0.121	-0.128	0.510	9.693	2.094	17.911	0.984
11	39.457	11.502	2.606	21.612	0.133	-0.141	0.561	11.701	2.539	21.574	0.980
12	43.044	13.689	3.101	25.720	0.145	-0.153	0.612	13.900	3.026	25.674	0.975
13	46.631	16.065	3.640	30.185	0.157	-0.166	0.664	16.288	3.557	30.132	0.971

Reports the components of the unconstrained fundamental law of portfolio enhancement over the OOS period (January 2015 to December 2024). The components of Equation 13 are forecast over each month. The mean is the average across the 120 months. L(5%) and U(95%) denote the lower and upper bounds of the 90% confidence interval based on the 5th and 95th percentiles. The active component mean is 0.066 with L(5%) and U(95%) 0.009 and 0.195 respectively. GTC^2 is the squared generalised transfer coefficient when $\psi = 0.01$.

To illustrate, for Australia, when the returns and CO2 intensity are equally weighted ($b = 1$), the average total enhancement of 0.173 consists of the active component of 0.066 (38%), the static component of 0.095 (55%) and the cross term component of 0.012 (7%). Similar

Table 11: USA: Maximum squared ER and its components

b	CO2 %	Static			Cross			Total			GTC^2
		Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.066	0.019	0.161	1.000
0.5	1.184	0.021	0.011	0.037	-0.001	-0.013	0.010	0.085	0.042	0.182	1.000
0.75	1.776	0.047	0.025	0.083	-0.002	-0.020	0.015	0.111	0.054	0.216	1.000
1	2.368	0.083	0.045	0.147	-0.003	-0.026	0.020	0.146	0.071	0.257	1.000
1.25	2.960	0.130	0.070	0.230	-0.004	-0.033	0.025	0.192	0.094	0.310	1.000
1.5	3.552	0.187	0.101	0.331	-0.004	-0.039	0.030	0.249	0.123	0.385	1.000
1.75	4.144	0.254	0.137	0.451	-0.005	-0.046	0.035	0.315	0.163	0.493	1.000
2	4.736	0.332	0.179	0.589	-0.006	-0.053	0.040	0.393	0.202	0.633	1.000
3	7.104	0.748	0.403	1.324	-0.009	-0.079	0.060	0.805	0.427	1.380	1.000
4	9.472	1.330	0.717	2.354	-0.012	-0.105	0.081	1.384	0.729	2.421	0.999
5	11.840	2.078	1.120	3.679	-0.015	-0.131	0.101	2.129	1.109	3.755	0.999
6	14.208	2.992	1.613	5.297	-0.018	-0.158	0.121	3.040	1.602	5.383	0.998
7	16.575	4.072	2.195	7.210	-0.021	-0.184	0.141	4.117	2.188	7.305	0.997
8	18.943	5.318	2.867	9.418	-0.024	-0.210	0.161	5.361	2.867	9.522	0.995
9	21.311	6.731	3.628	11.919	-0.027	-0.237	0.181	6.771	3.624	12.032	0.994
10	23.679	8.310	4.479	14.715	-0.030	-0.263	0.202	8.346	4.455	14.836	0.992
11	26.047	10.055	5.420	17.805	-0.033	-0.289	0.222	10.088	5.376	17.934	0.990
12	28.415	11.966	6.450	21.189	-0.036	-0.315	0.242	11.997	6.410	21.326	0.988
13	30.783	14.044	7.570	24.868	-0.039	-0.342	0.262	14.071	7.535	25.012	0.985

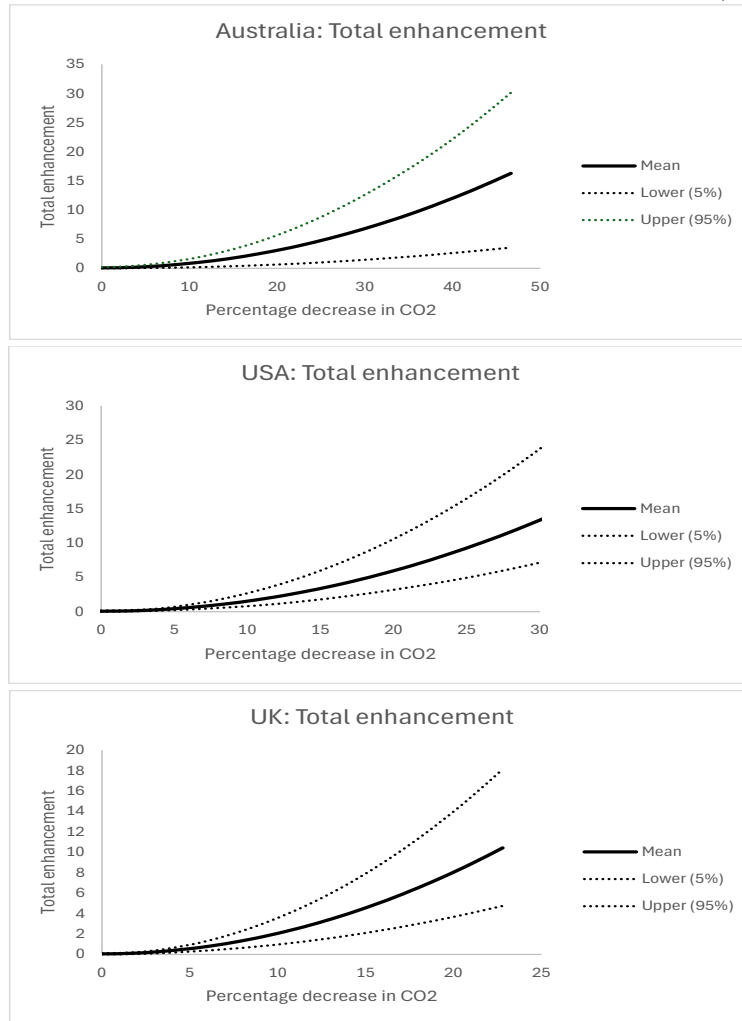
Reports the components of the unconstrained fundamental law of portfolio enhancement over the OOS period (January 2015 to December 2024). The components of Equation 13 are forecast over each month. The mean is the average across the 120 months. L(5%) and U(95%) denote the lower and upper bounds of the 90% confidence interval based on the 5th and 95th percentiles. The active component mean is 0.066 with L(5%) and U(95%) 0.019 and 0.161 respectively. GTC^2 is the squared generalised transfer coefficient when $\psi = 0.01$.

Table 12: UK: Maximum squared ER and its components

b	CO2 %	Static			Cross			Total			GTC^2
		Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	Mean	L(5%)	U(95%)	
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.052	0.003	0.124	1.000
0.5	0.877	0.015	0.007	0.027	0.003	-0.005	0.014	0.070	0.012	0.138	1.000
0.75	1.315	0.034	0.015	0.060	0.004	-0.008	0.021	0.090	0.023	0.163	1.000
1	1.753	0.061	0.027	0.106	0.005	-0.011	0.028	0.118	0.037	0.197	1.000
1.25	2.191	0.095	0.043	0.166	0.007	-0.014	0.035	0.154	0.057	0.250	1.000
1.5	2.630	0.137	0.061	0.239	0.008	-0.016	0.042	0.197	0.079	0.318	1.000
1.75	3.068	0.187	0.083	0.326	0.009	-0.019	0.049	0.248	0.108	0.402	1.000
2	3.506	0.244	0.109	0.426	0.011	-0.022	0.056	0.306	0.143	0.503	1.000
3	5.259	0.549	0.245	0.958	0.016	-0.033	0.084	0.617	0.302	1.031	1.000
4	7.012	0.976	0.435	1.702	0.021	-0.044	0.113	1.049	0.513	1.793	0.999
5	8.766	1.525	0.680	2.660	0.026	-0.054	0.141	1.603	0.762	2.773	0.998
6	10.519	2.196	0.979	3.830	0.032	-0.065	0.169	2.279	1.066	3.947	0.996
7	12.272	2.989	1.333	5.213	0.037	-0.076	0.197	3.078	1.430	5.335	0.995
8	14.025	3.904	1.741	6.809	0.042	-0.087	0.225	3.998	1.848	6.935	0.992
9	15.778	4.940	2.203	8.618	0.047	-0.098	0.253	5.040	2.320	8.748	0.989
10	17.531	6.099	2.720	10.639	0.053	-0.109	0.281	6.204	2.847	10.774	0.986
11	19.284	7.380	3.292	12.873	0.058	-0.120	0.310	7.490	3.428	13.013	0.983
12	21.037	8.783	3.917	15.320	0.063	-0.131	0.338	8.898	4.064	15.464	0.979
13	22.790	10.308	4.597	17.980	0.068	-0.142	0.366	10.428	4.754	18.129	0.974

Reports the components of the unconstrained fundamental law of portfolio enhancement over the OOS period (January 2015 to December 2024). The components of Equation 13 are forecast over each month. The mean is the average across the 120 months. L(5%) and U(95%) denote the lower and upper bounds of the 90% confidence interval based on the 5th and 95th percentiles. The active component mean is 0.049 with L(5%) and U(95%) 0.003 and 0.124 respectively. GTC^2 is the squared generalised transfer coefficient when $\psi = 0.01$.

Figure 6: Total enhancement versus CO2 reduction (%)



percentages apply for the other countries. The total enhancement for the USA consists of the active component 0.066 (45%), static component (57%) and the cross term -0.003 (-2%). For the UK the percentages are 44%, 52% and 4%.

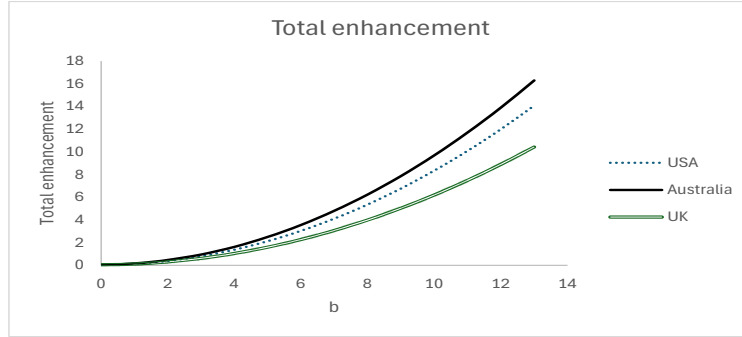
These portfolios (when $b = 1$) only achieve low percentage decreases in CO2 (3.6%, 2.4% and 1.8% respectively). Higher levels of CO2 reduction are achieved primarily through increases in the static component.

The plots in Figure 6 reveal that maximum squared ER versus the CO2 percentage reduction have a similar shape across countries. A 20% reduction in CO2 in Australia is achieved when $b \approx 6$ with an average squared maximum ER of ≈ 3.5 . The same level of CO2 percentage decrease is achieved in the USA when $b \approx 9$ with an average squared maximum ER of ≈ 6.8 and the UK when $b \approx 12$ with an average squared maximum ER of ≈ 8.8 . These results are consistent with the tracking error plots in section 3.1 which revealed that it is harder to achieve percentage reductions in CO2 in the UK.

In Figure 7 we plot the maximum squared ER versus the level of enhancement (b). The results mirror the TE results. The UK achieves the lowest level of enhancement given it has the lowest average cross-sectional coefficient of variation in CO2 (1.622). The USA has the second highest average coefficient of variation (1.790) and hence the second highest levels of enhancement. Australia on the other hand can achieve the highest level of enhancement which is consistent with Australian stocks having the highest coefficient of variation (2.309). The importance of the CV on portfolio enhancement is consistent with the weighted variance of the enhancement signal in the approximation expressed by equation 26.

Finally the GTCs for each country are close to unity and decrease as the level of enhancement (b) increases. Equivalent plots of the total enhancement for the constrained portfolios are therefore very similar to Figures 6 and 7. The small amount of drag on the unconstrained estimates is consistent with the small number of violations in the long only constraints and the low levels of turnover when performing the unconstrained optimizations.

Figure 7: Total enhancement versus b



6.4 Determinants of the Ex-Ante vs. Ex-Post Risk Gap

While the FLPE provides a rigorous *ex-ante* performance budget, a key empirical question remains: how well does the risk model predict realized volatility as the ambition of the mandate increases? In our results, we observe two distinct patterns regarding the "prediction gap" (the difference between realized and predicted tracking error).

First, we document that the prediction gap expands non-linearly as the enhancement preference b increases. For modest enhancement targets ($b \approx 1$), realized risk closely tracks the ex-ante model. However, for ambitious targets ($b > 2$), realized risk significantly exceeds the model's prediction.

This phenomenon is consistent with the principle of **maximization of estimation error** (Michaud 1989). As b increases, the optimizer seeks assets that offer the highest marginal enhancement per unit of predicted risk. These are invariably assets where the risk model underestimates the true covariance or volatility. By aggressively loading on these specific securities to satisfy a high b , the optimizer concentrates the portfolio in the "blind spots" of the risk model. Furthermore, aggressive tilting creates substantial exposure to a "Characteristic Factor" (e.g., a "Carbon Factor"). If this factor is not explicitly present in the risk model's factor structure, the model effectively treats this systematic risk as diversifiable idiosyncratic risk, leading to a structural underestimation of tracking error.

Second, and perhaps counter-intuitively, we observe that for a fixed level of b , tightening

the box constraints (e.g., reducing weight bounds) *does not* exacerbate this prediction gap, despite reducing the information ratio. We attribute this to the **regularizing effect of constraints** (Jagannathan and Ma 2003). Tight constraints act as a forcing function for diversification. To achieve a specific aggregate enhancement target under tight bounds, the optimizer is mathematically forced to utilize a broader set of assets (increasing effective N), rather than concentrating exposure in a few "best" names. This forced breadth diversifies the idiosyncratic estimation errors within the risk model. While constraints introduce a performance drag (lowering the GTC), they simultaneously improve the robustness of the risk forecast by preventing the optimizer from overfitting to specific assets where the model's precision signals are spuriously high.

The issue, is that by definition, optimized portfolios are 'optimal' ex-ante and assuming that the model characteristics hold in the true out of sample future. However, of course they will not (exactly) hold, and this can lead to very different ex-post outcomes.

In summary, increasing the intensity of the objective (b) exacerbates (ex-post) model risk through concentration, while tightening constraints mitigates (ex-post) model risk through forced diversification.

7 Conclusion

Active mandates increasingly mix two objectives: risk-adjusted value added from return forecasts and deterministic improvements in portfolio characteristics observed at the time of trading (for example, carbon intensity, exclusions, or factor tilts). This paper develops the Fundamental Law of Portfolio Enhancement (FLPE) as a unified performance budget for these hybrid, benchmark-relative problems. The FLPE is derived within tracking-error constrained quadratic optimization, built on the Ding and Martin (2017) factor-model representation of residual risk, and shown to nest the Fundamental Law of Active Management as the alpha-only special case.

The central output is an Enhancement Ratio that generalises the information ratio to objectives that combine return and characteristics. The maximised ratio decomposes into three economically interpretable components: an active term associated with alpha, a static term associated with deterministic characteristic tilts, and an interaction term that captures whether a characteristic requirement aligns with, or crowds out, the alpha portfolio. This decomposition provides a disciplined way to discuss trade-offs that are often treated informally, and it highlights when a characteristic mandate is likely to be "cheap" because it points in the same direction as alpha.

For purely deterministic objectives, the FLPE yields a practical rule of thumb for the tracking-error cost of characteristic improvement. Under simplifying assumptions used for intuition, attainable enhancement scales with tracking error and with three observable features of the opportunity set: effective breadth, average return precision, and the cross-sectional dispersion of the targeted characteristic.

Because implementable portfolios are constrained, the paper also provides a constraints-aware performance budget. We generalise the transfer-coefficient idea to hybrid objectives and use it as a diagnostic measure of "constraint drag" that separates unconstrained optimized benchmarks from feasible portfolios.

Empirically, rolling out-of-sample tests from 2015 to 2024 across large-cap universes in Australia, the United States, and the United Kingdom use carbon abatement as the deterministic characteristic and illustrate how the FLPE components map into realised enhancement-tracking-error trade-offs. The decomposition helps attribute cross-market differences to measurable inputs, including characteristic dispersion and the extent to which constraints bind. The evidence also motivates careful language about approximation quality: the FLPE is most informative as a first-order benchmark, and its accuracy varies with enhancement level and constraint tightness.

For practitioners, the implications are straightforward:

- Budget hybrid mandates coherently: translate characteristic targets into tracking-error budgets and compare the expected cost of the same target across universes using breadth, precision, and dispersion.
- Diagnose what is binding: use the generalised transfer coefficient and the interaction term to distinguish opportunity-set limitations from constraint-induced nonlinearity and from conflicts between alpha and the characteristic objective.
- Communicate curvature: when marginal costs rise sharply, reassess the target, the investable universe, and the constraints that are inducing nonlinearity—not only the tracking-error budget.

The proposed framework is critical for modern portfolio management, as it provides a quantitative tool for understanding the "performance drag" or "synergistic boost" that arises from imposing deterministic objectives on top of an active alpha model. It moves the discussion of such constraints from a qualitative trade-off to a quantitative component of the total performance budget.

Appendix A Derivation of the Maximized Enhancement Ratio

This appendix provides the complete derivation showing that the portfolio which maximizes mean-variance utility is the same portfolio that maximizes the squared Enhancement Ratio (ER^2), and proves that the maximized ER^2 value is $G_C = \boldsymbol{\mu}'_{obj,t} \Omega_c^{-1} \boldsymbol{\mu}_{obj,t}$.

For this derivation, we simplify the notation: $\boldsymbol{\mu} \equiv \boldsymbol{\mu}_{obj,t}$ and $\Omega \equiv \Omega_t$.

A.1 A.1: Optimal Portfolio (Utility Maximization)

We wish to find the active weights $\Delta \mathbf{w}$ that solve the constrained utility maximization problem:

$$\max_{\Delta \mathbf{w}} U(\Delta \mathbf{w}) = \Delta \mathbf{w}' \boldsymbol{\mu} - \lambda (\Delta \mathbf{w}' \Omega \Delta \mathbf{w}) \quad \text{s.t.} \quad \Delta \mathbf{w}' \mathbf{1} = 0 \quad (45)$$

We form the Lagrangian with multiplier ν :

$$\mathcal{L} = \Delta \mathbf{w}' \boldsymbol{\mu} - \lambda \Delta \mathbf{w}' \Omega \Delta \mathbf{w} - \nu (\Delta \mathbf{w}' \mathbf{1}) \quad (46)$$

Taking the partial derivative with respect to $\Delta \mathbf{w}$ and setting to zero:

$$\frac{\partial \mathcal{L}}{\partial \Delta \mathbf{w}} = \boldsymbol{\mu} - 2\lambda \Omega \Delta \mathbf{w} - \nu \mathbf{1} = \mathbf{0} \quad (47)$$

Solving for $\Delta \mathbf{w}$:

$$\Delta \mathbf{w} = \frac{1}{2\lambda} \Omega^{-1} (\boldsymbol{\mu} - \nu \mathbf{1}) \quad (48)$$

Now, we use the constraint $\Delta \mathbf{w}' \mathbf{1} = 0$ to solve for ν :

$$\left(\frac{1}{2\lambda} \Omega^{-1} (\boldsymbol{\mu} - \nu \mathbf{1}) \right)' \mathbf{1} = 0 \quad \implies \quad (\boldsymbol{\mu}' - \nu \mathbf{1}') \Omega^{-1} \mathbf{1} = 0 \quad (49)$$

$$\boldsymbol{\mu}'\Omega^{-1}\mathbf{1} - \nu(\mathbf{1}'\Omega^{-1}\mathbf{1}) = 0 \quad \implies \quad \nu = \frac{\boldsymbol{\mu}'\Omega^{-1}\mathbf{1}}{\mathbf{1}'\Omega^{-1}\mathbf{1}} \quad (50)$$

Substituting ν back into the equation for $\Delta\mathbf{w}$:

$$\Delta\mathbf{w}^* = \frac{1}{2\lambda} \left(\Omega^{-1}\boldsymbol{\mu} - \Omega^{-1}\mathbf{1} \left(\frac{\boldsymbol{\mu}'\Omega^{-1}\mathbf{1}}{\mathbf{1}'\Omega^{-1}\mathbf{1}} \right) \right) \quad (51)$$

Factoring out $\boldsymbol{\mu}$ (and using $\boldsymbol{\mu}'\Omega^{-1}\mathbf{1} = \mathbf{1}'\Omega^{-1}\boldsymbol{\mu}$ since it's a scalar):

$$\Delta\mathbf{w}^* = \frac{1}{2\lambda} \left(\Omega^{-1} - \frac{\Omega^{-1}\mathbf{1}\mathbf{1}'\Omega^{-1}}{\mathbf{1}'\Omega^{-1}\mathbf{1}} \right) \boldsymbol{\mu} \quad (52)$$

We define the matrix in the parentheses as the **constrained inverse**, Ω_c^{-1} .

$$\Delta\mathbf{w}^* = \frac{1}{2\lambda} \Omega_c^{-1} \boldsymbol{\mu} \quad (53)$$

This is the optimal portfolio that maximizes utility.

A.2 A.2: ER^2 Maximization

Now we solve a different problem: find the portfolio $\Delta\mathbf{w}_{ER}$ that maximizes the squared Enhancement Ratio, $G(\Delta\mathbf{w})$.

$$\max_{\Delta\mathbf{w}_{ER}} G(\Delta\mathbf{w}_{ER}) = \frac{(\Delta\mathbf{w}_{ER}'\boldsymbol{\mu})^2}{\Delta\mathbf{w}_{ER}'\Omega\Delta\mathbf{w}_{ER}} \quad \text{s.t.} \quad \Delta\mathbf{w}_{ER}'\mathbf{1} = 0 \quad (54)$$

Since G is invariant to scaling (i.e., $G(k\Delta\mathbf{w}) = G(\Delta\mathbf{w})$), we can re-frame this as maximizing the numerator subject to the denominator being 1 (and the original constraint).

$$\max_{\Delta\mathbf{w}_{ER}} \Delta\mathbf{w}_{ER}'\boldsymbol{\mu} \quad \text{s.t.} \quad \Delta\mathbf{w}_{ER}'\Omega\Delta\mathbf{w}_{ER} = 1 \quad \text{and} \quad \Delta\mathbf{w}_{ER}'\mathbf{1} = 0 \quad (55)$$

The Lagrangian with multipliers ν_1 and ν_2 is:

$$\mathcal{L}_{ER} = \Delta \mathbf{w}'_{ER} \boldsymbol{\mu} - \nu_1 (\Delta \mathbf{w}'_{ER} \Omega \Delta \mathbf{w}_{ER} - 1) - \nu_2 (\Delta \mathbf{w}'_{ER} \mathbf{1}) \quad (56)$$

Taking the derivative and setting to zero:

$$\frac{\partial \mathcal{L}_{ER}}{\partial \Delta \mathbf{w}_{ER}} = \boldsymbol{\mu} - 2\nu_1 \Omega \Delta \mathbf{w}_{ER} - \nu_2 \mathbf{1} = \mathbf{0} \quad (57)$$

Solving for $\Delta \mathbf{w}_{ER}$:

$$\Delta \mathbf{w}_{ER} = \frac{1}{2\nu_1} \Omega^{-1} (\boldsymbol{\mu} - \nu_2 \mathbf{1}) \quad (58)$$

Notice this equation has the *exact same algebraic form* as the utility-maximizing portfolio in equation 48. The unknown scalar $\frac{1}{2\lambda}$ is simply replaced by a different unknown scalar $\frac{1}{2\nu_1}$, and the unknown multiplier ν is replaced by ν_2 .

To solve for ν_2 , we follow the identical procedure as in A.1, using the constraint $\Delta \mathbf{w}'_{ER} \mathbf{1} = 0$:

$$\left(\frac{1}{2\nu_1} \Omega^{-1} (\boldsymbol{\mu} - \nu_2 \mathbf{1}) \right)' \mathbf{1} = 0 \quad (59)$$

The scalar $\frac{1}{2\nu_1}$ is non-zero and drops out:

$$(\boldsymbol{\mu}' - \nu_2 \mathbf{1}') \Omega^{-1} \mathbf{1} = 0 \quad (60)$$

$$\nu_2 = \frac{\boldsymbol{\mu}' \Omega^{-1} \mathbf{1}}{\mathbf{1}' \Omega^{-1} \mathbf{1}} \quad (61)$$

This is the *exact same expression* as we found for ν in equation 50. Now we substitute this ν_2 back into our equation for $\Delta \mathbf{w}_{ER}$ (equation 58):

$$\Delta \mathbf{w}_{ER}^* = \frac{1}{2\nu_1} \left(\Omega^{-1} \boldsymbol{\mu} - \Omega^{-1} \mathbf{1} \left(\frac{\boldsymbol{\mu}' \Omega^{-1} \mathbf{1}}{\mathbf{1}' \Omega^{-1} \mathbf{1}} \right) \right) \quad (62)$$

Factoring out $\boldsymbol{\mu}$ as before:

$$\Delta \mathbf{w}_{ER}^* = \frac{1}{2\nu_1} \left(\Omega^{-1} - \frac{\Omega^{-1} \mathbf{1} \mathbf{1}' \Omega^{-1}}{\mathbf{1}' \Omega^{-1} \mathbf{1}} \right) \boldsymbol{\mu} \quad (63)$$

Using our definition of the constrained inverse Ω_c^{-1} :

$$\Delta \mathbf{w}_{ER}^* = \frac{1}{2\nu_1} \Omega_c^{-1} \boldsymbol{\mu} \quad (64)$$

Comparing this to the utility-maximizing portfolio $\Delta \mathbf{w}^* = \frac{1}{2\lambda} \Omega_c^{-1} \boldsymbol{\mu}$ (equation 53), we see they are identical in direction and differ only by a scalar constant. Therefore, $\Delta \mathbf{w}_{ER}^* \propto \Delta \mathbf{w}^*$. The portfolio that maximizes utility is, up to a scalar, the same portfolio that maximizes the Enhancement Ratio.

A.3 A.3: Deriving the Maximized G_C Value

Finally, we prove the identity $G_C = \boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu}$. We take the maximizing portfolio $\Delta \mathbf{w}_{ER}^* = k \Omega_c^{-1} \boldsymbol{\mu}$ (for some scalar $k \neq 0$) and substitute it into the ER^2 formula G_C :

$$G_C = \frac{(\Delta \mathbf{w}_{ER}^* \boldsymbol{\mu})^2}{\Delta \mathbf{w}_{ER}^{*'} \Omega \Delta \mathbf{w}_{ER}^*} = \frac{(k \boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu})^2}{(k \Omega_c^{-1} \boldsymbol{\mu})' \Omega (k \Omega_c^{-1} \boldsymbol{\mu})} \quad (65)$$

The scalar k^2 cancels from the numerator and denominator:

$$G_C = \frac{(\boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu})^2}{\boldsymbol{\mu}' \Omega_c^{-1} \Omega \Omega_c^{-1} \boldsymbol{\mu}} \quad (66)$$

We now use the key property of this constrained inverse operator, which is proven in **Appendix C**: $\Omega_c^{-1} \Omega \Omega_c^{-1} = \Omega_c^{-1}$. This property holds because Ω_c^{-1} acts as a generalized inverse for the covariance matrix Ω within the constrained (sum-to-zero) subspace. Substituting this identity into the denominator:

$$G_C = \frac{(\boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu})^2}{\boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu}} \quad (67)$$

This simplifies to:

$$G_C = \boldsymbol{\mu}' \Omega_c^{-1} \boldsymbol{\mu} \quad (68)$$

This completes the derivation. The ex-ante FLPE is the expectation of this quantity.

$$E[G_C] = E[\boldsymbol{\mu}'_{obj,t} \Omega_c^{-1} \boldsymbol{\mu}_{obj,t}] \quad (69)$$

Appendix B Derivation of the Woodbury Matrix Identity Form

In Section F, we used the matrix identity:

$$V(UCV + A)^{-1}U = (C + (VA^{-1}U)^{-1})^{-1} \quad (70)$$

This is a specific form of the more general Woodbury Matrix Identity, often called the Sherman-Morrison-Woodbury formula (Henderson and Searle 1981). We prove it here.

1. The Standard Woodbury Identity: The standard identity is:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \quad (71)$$

2. The Target Identity: We want to prove $V(UCV + A)^{-1}U = (C + (VA^{-1}U)^{-1})^{-1}$.

3. Proof: We start with the LHS, $V(UCV + A)^{-1}U$. We substitute the standard identity (equation 71) for the inverse term:

$$LHS = V [A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}] U \quad (72)$$

Distribute V (from the left) and U (from the right):

$$LHS = VA^{-1}U - (VA^{-1}U)(C^{-1} + VA^{-1}U)^{-1}(VA^{-1}U) \quad (73)$$

To simplify, let $K = VA^{-1}U$ and $M = C^{-1}$.

$$LHS = K - K(M + K)^{-1}K \quad (74)$$

Factor out K from the left:

$$LHS = K [I - (M + K)^{-1}K] \quad (75)$$

Find a common denominator for the bracketed term by pre-multiplying I by $(M + K)^{-1}(M + K)$:

$$LHS = K [(M + K)^{-1}(M + K) - (M + K)^{-1}K] \quad (76)$$

$$= K(M + K)^{-1} [(M + K) - K] \quad (77)$$

$$= K(M + K)^{-1}M \quad (78)$$

Now we have $LHS = K(M + K)^{-1}M$. We use the identity $X(Y + X)^{-1}Y = (X^{-1} + Y^{-1})^{-1}$.

Let $X = K$ and $Y = M$.

$$LHS = (K^{-1} + M^{-1})^{-1} \quad (79)$$

Substitute back our original terms:

- $K^{-1} = (VA^{-1}U)^{-1}$
- $M^{-1} = (C^{-1})^{-1} = C$

Therefore:

$$LHS = ((VA^{-1}U)^{-1} + C)^{-1} \quad (80)$$

This is the RHS of the identity we wanted to prove (equation 70). Q.E.D.

Appendix C Proof of the Generalized Inverse Property

Here we provide the step-by-step proof for the property $\Omega_c^{-1}\Omega\Omega_c^{-1} = \Omega_c^{-1}$, which was used in Appendix A.3.

To simplify the algebra, let $A = \Omega^{-1}$ (which implies $A^{-1} = \Omega$) and let $\mathbf{v} = \mathbf{1}$.

The constrained inverse from Appendix A.1 is:

$$\Omega_c^{-1} = A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \quad (81)$$

We want to prove that the Left Hand Side (LHS) = Ω_c^{-1} .

$$LHS = (\Omega_c^{-1})\Omega(\Omega_c^{-1}) = \left(A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) A^{-1} \left(A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \quad (82)$$

Step 1: Distribute A^{-1} into the second term.

$$LHS = \left(A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \left(A^{-1}A - A^{-1}\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \quad (83)$$

$$= \left(A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \left(I - \frac{\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \quad (84)$$

Step 2: Expand the two resulting binomials (a 4-term expansion).

$$LHS = \underbrace{A \cdot I}_{\text{Term 1}} - \underbrace{A \cdot \frac{\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}}_{\text{Term 2}} \quad (85)$$

$$- \underbrace{\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \cdot I}_{\text{Term 3}} + \underbrace{\left(\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right) \left(\frac{\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}}\right)}_{\text{Term 4}} \quad (86)$$

Step 3: Simplify each term. Terms 1, 2, and 3 are already simple. We focus on Term 4.

$$\text{Term 4} = + \frac{(A\mathbf{v}\mathbf{v}'A)(\mathbf{v}\mathbf{v}'A)}{(\mathbf{v}'A\mathbf{v})^2} \quad (87)$$

Note that $\mathbf{v}'A\mathbf{v}$ is a 1×1 scalar. We can group this scalar in the numerator.

$$\text{Term 4} = + \frac{A\mathbf{v}(\mathbf{v}'A\mathbf{v})\mathbf{v}'A}{(\mathbf{v}'A\mathbf{v})^2} \quad (88)$$

The scalar $(\mathbf{v}'A\mathbf{v})$ in the numerator and denominator cancels:

$$\text{Term 4} = + \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \quad (89)$$

Step 4: Sum the four simplified terms.

$$LHS = (A) - \left(\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \right) - \left(\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \right) + \left(\frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \right) \quad (90)$$

$$= A - \frac{A\mathbf{v}\mathbf{v}'A}{\mathbf{v}'A\mathbf{v}} \quad (91)$$

This final expression is, by definition, Ω_c^{-1} .

$$LHS = \Omega_c^{-1} = RHS \quad (92)$$

The proof is complete.

Appendix D Special Case: Diagonal Risk Model

D.1 D.1: Static Contribution with Diagonal Risk (Constrained)

We first analyze the Static Contribution, $G_{\text{Static}} = b^2 E[\mathbf{d}'_t \Omega_c^{-1} \mathbf{d}_t]$, using the full constrained inverse Ω_c^{-1} . From Appendix A, we have:

$$G_{\text{Static}}/b^2 = \mathbf{d}'_t \Omega_t^{-1} \mathbf{d}_t - \frac{(\mathbf{d}'_t \Omega_t^{-1} \mathbf{1})^2}{\mathbf{1}' \Omega_t^{-1} \mathbf{1}} \quad (93)$$

We evaluate each piece using our diagonal assumption:

- $\mathbf{d}'_t \Omega_t^{-1} \mathbf{d}_t = \sum_{i=1}^N d_{t,i}(p_i) d_{t,i} = \sum_{i=1}^N p_i d_{t,i}^2$
- $\mathbf{d}'_t \Omega_t^{-1} \mathbf{1} = \sum_{i=1}^N d_{t,i}(p_i)(1) = \sum_{i=1}^N p_i d_{t,i}$
- $\mathbf{1}' \Omega_t^{-1} \mathbf{1} = \sum_{i=1}^N (1)(p_i)(1) = \sum_{i=1}^N p_i$

Substituting these back in:

$$G_{\text{Static}}/b^2 = \sum_{i=1}^N p_i d_{t,i}^2 - \frac{(\sum_{i=1}^N p_i d_{t,i})^2}{\sum_{i=1}^N p_i} \quad (94)$$

This expression is a well-known statistical identity. The precision-weighted variance of $\mathbf{d}_t$ is:

$$\text{Var}_p(\mathbf{d}_t) = E_p[\mathbf{d}_t^2] - (E_p[\mathbf{d}_t])^2 = \left(\frac{\sum p_i d_{t,i}^2}{\sum p_i} \right) - \left(\frac{\sum p_i d_{t,i}}{\sum p_i} \right)^2 \quad (95)$$

If we multiply this by the total precision, $\sum p_i$:

$$\left(\sum_{i=1}^N p_i \right) \text{Var}_p(\mathbf{d}_t) = \sum_{i=1}^N p_i d_{t,i}^2 - \frac{(\sum_{i=1}^N p_i d_{t,i})^2}{\sum_{i=1}^N p_i} \quad (96)$$

This is identical to our expression for G_{Static}/b^2 . Therefore, for a diagonal risk model with the constraint:

$$G_{\text{Static}} = b^2 \left(\sum_{i=1}^N p_i \right) \text{Var}_p(\mathbf{d}_t) = b^2 \left(\sum_{i=1}^N \frac{1}{\sigma_i^2} \right) \text{Var}_p(\mathbf{d}_t) = b^2 N \bar{\mathbf{p}} \text{Var}_p(\mathbf{d}_t) \quad (97)$$

This shows that the static contribution is directly proportional to the total precision of the universe and the precision-weighted variance of the deterministic characteristic.

Appendix E Proof that GTC is equal to the correlation of the active return streams

We now derive the formula for $\text{Corr}(r_c, r_u)$ from first principles.

1. **Start with the definition of correlation:**

$$\text{Corr}(r_c, r_u) = \frac{\text{Cov}(r_c, r_u)}{\sqrt{\text{Var}(r_c)}\sqrt{\text{Var}(r_u)}} \quad (98)$$

2. **Analyze the Denominator (Variances):** The variance of the constrained portfolio's return stream is:

$$\text{Var}(r_c) = \text{Var}(\mathbf{h}'_c \mathbf{r}_t) = \mathbf{h}'_c \text{Cov}(\mathbf{r}_t) \mathbf{h}_c = \mathbf{h}'_c \Omega_t \mathbf{h}_c \quad (99)$$

By analogy, the variance of the unconstrained portfolio's return stream is:

$$\text{Var}(r_u) = \text{Var}(\mathbf{h}'_u \mathbf{r}_t) = \mathbf{h}'_u \Omega_t \mathbf{h}_u \quad (100)$$

3. **Analyze the Numerator (Covariance):** The covariance between the two return streams is:

$$\text{Cov}(r_c, r_u) = \text{Cov}(\mathbf{h}'_c \mathbf{r}_t, \mathbf{h}'_u \mathbf{r}_t) = \mathbf{h}'_c \text{Cov}(\mathbf{r}_t) \mathbf{h}_u = \mathbf{h}'_c \Omega_t \mathbf{h}_u \quad (101)$$

4. **Substitute $\mathbf{h}_u$ into the Numerator:** From Section 5.1, we know $\mathbf{h}_u = k\Omega_t^{-1}\boldsymbol{\mu}_{obj}$ for some scalar k .

$$\text{Numerator} = \mathbf{h}'_c \Omega_t (k\Omega_t^{-1}\boldsymbol{\mu}_{obj}) = k(\mathbf{h}'_c \Omega_t \Omega_t^{-1} \boldsymbol{\mu}_{obj}) = k(\mathbf{h}'_c \boldsymbol{\mu}_{obj}) \quad (102)$$

5. **Substitute $\mathbf{h}_u$ into the Denominator (Part 2):**

$$\sqrt{\text{Var}(r_u)} = \sqrt{\mathbf{h}'_u \Omega_t \mathbf{h}_u} = \sqrt{(k\Omega_t^{-1} \boldsymbol{\mu}_{obj})' \Omega_t (k\Omega_t^{-1} \boldsymbol{\mu}_{obj})} \quad (103)$$

$$= \sqrt{k^2 \boldsymbol{\mu}'_{obj} (\Omega_t^{-1})' \Omega_t \Omega_t^{-1} \boldsymbol{\mu}_{obj}} \quad (104)$$

$$= \sqrt{k^2 \boldsymbol{\mu}'_{obj} \Omega_t^{-1} I \boldsymbol{\mu}_{obj}} \quad (\text{since } \Omega_t^{-1} \text{ is symmetric}) \quad (105)$$

$$= \sqrt{k^2 (\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj})} = k \sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}} \quad (106)$$

6. **Re-assemble the Correlation:**

$$\text{Corr}(r_c, r_u) = \frac{k(\mathbf{h}'_c \boldsymbol{\mu}_{obj})}{\sqrt{\mathbf{h}'_c \Omega_t \mathbf{h}_c} \cdot (k \sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}})} \quad (107)$$

The scaling constant k cancels, leaving:

$$\text{Corr}(r_c, r_u) = \frac{\mathbf{h}'_c \boldsymbol{\mu}_{obj}}{\sqrt{\mathbf{h}'_c \Omega_t \mathbf{h}_c} \sqrt{\boldsymbol{\mu}'_{obj} \Omega_t^{-1} \boldsymbol{\mu}_{obj}}} \quad (108)$$

This final expression is identical to our definition of the *GTC*. Q.E.D.

E.1 Deriving the Constrained FLPE

1. **Square the GTC definition from equation (22):**

$$GTC^2 = \left(\frac{ER_C}{ER_U} \right)^2 = \frac{ER_C^2}{ER_U^2} \quad (109)$$

2. **Recognize the terms:** ER_C^2 is G_C , the squared ER of the constrained portfolio.

ER_U^2 is G_U , the squared ER of the unconstrained portfolio.

$$GTC^2 = \frac{G_C}{G_U} \quad (110)$$

3. Solve for the Constrained ER^2 :

$$G_C = GTC^2 \cdot G_U \tag{111}$$

This proves that the squared constrained ER is simply the squared unconstrained ER multiplied by the squared GTC .

Appendix F Derivation Detail (Active Contribution)

We want to simplify $G_{\text{Active, U}} = a^2 \alpha'_t \Omega_t^{-1} \alpha_t$, ignoring the expectation for a moment.

1. **Substitute definitions:** $\alpha_t = \Lambda_t^{1/2} Z_{t-1} \mathbf{f}$ (Ding and Martin 2017, equation 37) and $\Omega_t = \Lambda_t^{1/2} (Z_{t-1} \Sigma_{\mathbf{f}} Z'_{t-1} + \sigma_\epsilon^2 I) \Lambda_t^{1/2}$ (Ding and Martin 2017, equation 38).

$$G_{\text{Active, U}}/a^2 = (\Lambda_t^{1/2} Z_{t-1} \mathbf{f})' \left[\Lambda_t^{1/2} (Z_{t-1} \Sigma_{\mathbf{f}} Z'_{t-1} + \sigma_\epsilon^2 I) \Lambda_t^{1/2} \right]^{-1} (\Lambda_t^{1/2} Z_{t-1} \mathbf{f}) \quad (112)$$

2. **Cancel Λ_t terms:** The $\Lambda_t^{1/2}$ terms cancel, leaving:

$$G_{\text{Active, U}}/a^2 = \mathbf{f}' Z'_{t-1} [Z_{t-1} \Sigma_{\mathbf{f}} Z'_{t-1} + \sigma_\epsilon^2 I]^{-1} Z_{t-1} \mathbf{f} \quad (113)$$

3. **Apply Matrix Identity:** We apply a specific form of the Woodbury Matrix Identity, which is proven in **Appendix B**:

$$V(UCV + A)^{-1}U = (C + (VA^{-1}U)^{-1})^{-1} \quad (114)$$

4. **Set terms for the identity:** Let:

- $A = \sigma_\epsilon^2 I$ (an $N \times N$ matrix)
- $U = Z_{t-1}$ (an $N \times K$ matrix)
- $C = \Sigma_{\mathbf{f}}$ (a $K \times K$ matrix)
- $V = Z'_{t-1}$ (a $K \times N$ matrix)

Our inner block is $Z'_{t-1} (Z_{t-1} \Sigma_{\mathbf{f}} Z'_{t-1} + \sigma_\epsilon^2 I)^{-1} Z_{t-1}$, which matches the form $V(UCV + A)^{-1}U$.

5. **Apply the identity:** The inner block simplifies to $(C + (VA^{-1}U)^{-1})^{-1}$. Let's find the components:

- $C = \Sigma_{\mathbf{f}}$
- $VA^{-1}U = Z'_{t-1}(\sigma_{\epsilon}^2 I)^{-1}Z_{t-1} = \frac{1}{\sigma_{\epsilon}^2}Z'_{t-1}Z_{t-1}$
- $(VA^{-1}U)^{-1} = (\frac{1}{\sigma_{\epsilon}^2}Z'_{t-1}Z_{t-1})^{-1} = \sigma_{\epsilon}^2(Z'_{t-1}Z_{t-1})^{-1}$

So, the inner block is $(\Sigma_{\mathbf{f}} + \sigma_{\epsilon}^2(Z'_{t-1}Z_{t-1})^{-1})^{-1}$.

6. Substitute back and apply Large-N (number of stocks) Approximation:

$$G_{\text{Active, U}}/a^2 = \mathbf{f}' (\Sigma_{\mathbf{f}} + \sigma_{\epsilon}^2(Z'_{t-1}Z_{t-1})^{-1})^{-1} \mathbf{f} \quad (115)$$

Let $A_N = \frac{1}{N}Z'_{t-1}Z_{t-1}$. Thus, $(Z'_{t-1}Z_{t-1})^{-1} = \frac{1}{N}A_N^{-1}$.

$$G_{\text{Active, U}}/a^2 = \mathbf{f}' \left(\Sigma_{\mathbf{f}} + \frac{\sigma_{\epsilon}^2}{N}A_N^{-1} \right)^{-1} \mathbf{f} \quad (116)$$

Taking the expectation, and using the large-N approximation $E[A_N^{-1}] \approx E[A_N]^{-1} \approx I^{-1} = I$:

$$E[G_{\text{Active, U}}] \approx a^2 \mathbf{f}' \left(\Sigma_{\mathbf{f}} + \frac{\sigma_{\epsilon}^2}{N}I \right)^{-1} \mathbf{f} \quad (117)$$

This is the square of the factor-model expression for the information ratio formula from (Ding and Martin 2017, equation 39).

Appendix G Derivation Detail (Term 3: The Cross-Term Contribution)

We substitute $\alpha_t = \Lambda_t^{1/2}Z_{t-1}\mathbf{f}$ (Ding and Martin 2017, equation 37) and $\mathbf{d}_t = X_{t-1}\mathbf{c}_d$:

$$E[G_{\text{Cross, U}}] = 2abE \left[(\Lambda_t^{1/2}Z_{t-1}\mathbf{f})' \Omega_t^{-1} (X_{t-1}\mathbf{c}_d) \right] \quad (118)$$

Since $\mathbf{f}$ and $\mathbf{c}_d$ are constant vectors, we can pull them out of the expectation:

$$E[G_{\text{Cross, U}}] = 2ab\mathbf{f}' \left(E[Z'_{t-1}\Lambda_t^{1/2}\Omega_t^{-1}X_{t-1}] \right) \mathbf{c}_d \quad (119)$$

To make this interpretable, let us define the $K \times M$ matrix $\Sigma_{Z\Omega X}$ as:

$$\Sigma_{Z\Omega X} \equiv E[Z'_{t-1}\Lambda_t^{1/2}\Omega_t^{-1}X_{t-1}] \quad (120)$$

The cross-term is therefore:

$$E[G_{\text{Cross, U}}] = 2ab\mathbf{f}'\Sigma_{Z\Omega X}\mathbf{c}_d \quad (121)$$

Appendix H Positivity of $\mathcal{Q} = \mathbf{z}'(\Lambda_t^{1/2}\Omega_t^{-1})\mathbf{z}$

Let $W = \Lambda_t^{1/2}\Omega_t^{-1}$. We must determine if the quadratic form $\mathbf{z}'W\mathbf{z}$ is always positive.

- $\Lambda_t^{1/2}$ is a **Diagonal** matrix with positive entries (volatilities).
- Ω_t^{-1} is a **Symmetric Positive Definite (SPD)** matrix (inverse covariance).

Since the product of a diagonal matrix and a symmetric matrix is generally *not* symmetric, W is not symmetric. However, a fundamental property of linear algebra states that the value of any quadratic form $\mathbf{z}'W\mathbf{z}$ is determined solely by the **symmetric part** of the matrix, W_S :

$$\mathbf{z}'W\mathbf{z} = \mathbf{z}'\left(\frac{W + W'}{2}\right)\mathbf{z} = \mathbf{z}'W_S\mathbf{z} \quad (122)$$

(The skew-symmetric part, $\frac{W-W'}{2}$, contributes exactly zero to the scalar sum).

Therefore, the cross-term contributes positively if and only if the symmetric part W_S is positive definite. While it is mathematically possible to construct pathological examples where this does not hold (e.g., extreme volatilities paired with high negative correlations), in standard financial markets, the alignment between variances and covariances ensures that W_S remains positive definite. Thus, we assume $\mathbf{z}'W\mathbf{z} > 0$.

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